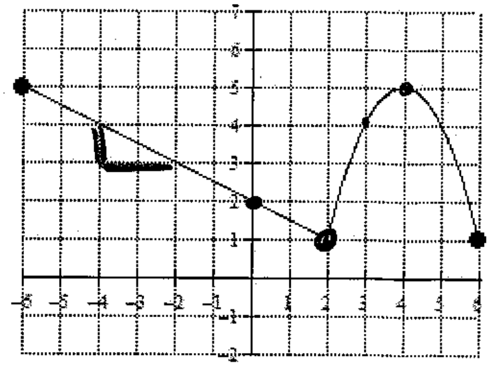
**AP Calculus AB**
**Unit 3 – Day 5 – Assignment**

**Name:** Answer Key*



Use the graph of $H(x)$, pictured to the right, to complete exercises 1 – 4.

1. The graph of $H(x)$ is continuous on its domain but not differentiable at all values on its domain. At what value on $-6 < x < 6$ is $H(x)$ not differentiable. Give a graphical reason for your answer.

$x=2$ is a value at which $H(x)$ is continuous but not differentiable b/c the graph has a cusp.

2. Write an equation of $H(x)$ and show analytically that $H(x)$ is, in fact, continuous at the $x$ – value that you identified in exercise 1. Show and explain your work.

y-int = 2
slope = $-1/2$

$x^2$
refl.
up 5,
Right 4

$$H(x)=\begin{cases} -\frac{1}{2}x+2, & -6\le x\le 2 \\ -(x-4)^2+5, & 2< x\le 6\end{cases}$$

① $H(2) = -\frac{1}{2}(2)+2 = -1+2=1$, $H(2)$ is defined ✓

② $\lim\limits_{x\to 2^-} H(x) = -\frac{1}{2}(2)+2 = 1$, $\lim\limits_{x\to 2^+} H(x) = -(2-4)^2+5 = 1$

$\lim\limits_{x\to 2} H(x)$ exists ✓

③ $H(2) = \lim\limits_{x\to 2} H(x) = 1$ ✓

$\therefore H(x)$ is cont. at $x=2$.

3. Show analytically that $H(x)$ is, in fact, not differentiable at the $x$ – value that you identified in exercise 1. Show and explain your work.

$$H'(x)=\begin{cases} -\frac{1}{2}, & -6\le x\le 2 \\ -2(x-4), & 2< x\le 6\end{cases}$$

① $H(x)$ is cont at $x=2$ (see #2)

② $\lim\limits_{x\to 2^-} H'(x) = -1/2$ $\quad \lim\limits_{x\to 2^+} H(x) = -2(2-4) = 4$

Since $\lim\limits_{x\to 2^-} H'(x) \ne \lim\limits_{x\to 2^+} H'(x)$, then $H(x)$ is not diff. at $x=2$.

4. Given the graph of $H(x)$ pictured above, find the equation of the tangent line to the graph of $P(x)=\sqrt{H(x)}$ when $x=3$.

P.O.T →
$P(3) = \sqrt{H(3)} = \sqrt{4} = 2$ $\quad (3,2)$

use eqn in #3

S.O.T →
$P(x) = [H(x)]^{1/2}$

$P'(x) = \frac{1}{2}[H(x)]^{-1/2}[H'(x)]$

$= \frac{1}{2\sqrt{H(x)}}\cdot H'(x)$

$P'(3) = \frac{1}{2\sqrt{H(3)}}\cdot H'(3)$

$= \frac{1}{2\sqrt{4}}\cdot -2(3-4)$

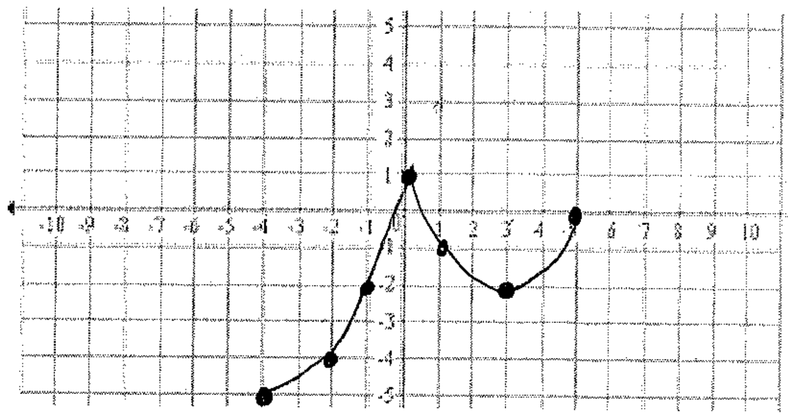
$= \frac{1}{4}\cdot(2) = \frac{1}{2}$

$\boxed{y-2=\frac{1}{2}(x-3)}$

A continuous function on the interval $-4 < x < 5$, $h(x)$, is described in the table below. Use the information to complete exercises 5 – 8.

| $x$ | $-4$ | $-2$ | $-1$ | $0$ | $-4 < x < 0$ | $1$ | $3$ | $0 < x < 3$ | $3 < x < 5$ | $5$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $h(x)$ | $-5$ | $-4$ | $-2$ | $1$ | Increasing & Concave Up | $-1$ | $-2$ | Decreasing & Concave Up | Increasing & Concave Up | $0$ |

5. Sketch a graph of $h(x)$.



6. Estimate the value of $h'(-2)$. Does this value support the claim that $h(x)$ is increasing on the interval $-4 < x < 0$? Give a reason for your answer.

$h'(-2)$ $(-4,-5)$ $(-1,-2)$

$m = \frac{-2-(-5)}{-1-(-4)} = \frac{3}{3} = 1$

Since $h'(-2) > 0$, then yes, $h(x)$ is increasing on the interval $-4 < x < 0$.

7. There are three $x$ – values in the domain of $h$ at which $h(x)$ is not differentiable. What are these three values and give a reason for why $h(x)$ is not differentiable at these values.

$X = -4$ → H(x) is not continuous at $x = -4$

$X = 0$ → H(x) has a cusp at $x = 0$.

$X = 5$ → H(x) is not continuous at $x = 5$

8. On what interval(s) of $x$ is $h'(x) > 0$? Give a reason for your answer.

$h'(x) > 0$ when $h(x)$ is increasing.

$\boxed{(-4,0) \cup (3,5)}$

$f'(x) = 1$

9. At what value(s) of $x$ will the graph of $f(x) = 2e^{2x} - 3x$ have a tangent line whose slope is 1?

$f'(x) = 2e^{2x}(2) - 3 = 4e^{2x} - 3$

$4e^{2x} - 3 = 1$

$4e^{2x} = 4$

$e^{2x} = 1$

$\ln e^{2x} = \ln 1 \rightarrow 2x = 0$

$\boxed{x = 0}$

10. The graph of $x - 2y = 9$ is parallel to the normal line to the graph of $f(x)$ when $x = 5$. What is the value of $f'(5)$? Justify your answer.

$x - 2y = 9$

$-2y = -x + 9$

$y = \frac{1}{2}x - \frac{9}{2}$

slope of normal $= 1/2$ $\rightarrow$ $f'(5) =$ slope of tangent $= \boxed{-2}$

11. Let $f$ be defined by the function $f(x) = \begin{cases} 3 - x, & x < 1 \\ ax^2 + bx, & x \geq 1 \end{cases}$.

a. If the function is continuous at $x = 1$, what is the relationship between $a$ and $b$? Explain your reasoning using limits.

① $f(1)$ is defined ✓

② For $f(x)$ to be cont. at $x=1$ $\rightarrow$ then $\lim_{x\to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$3 - 1 = a(1)^2 + b(1)$

$2 = a + b$

b. Find the unique values of $a$ and $b$ that will make $f$ both continuous and differentiable at $x = 1$. Show your analysis using limits.

For $f(x)$ to be diff. at $x=1$, $\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^+} f'(x)$.

$f'(x) = \begin{cases} -1, & x<1 \\ 2ax + b, & x \geq 1 \end{cases}$

$-1 = 2a(1) + b$

$-1 = 2a + b$

$a + b = 2$

$\ominus 2a \ominus b = \oplus 1$

$-a = 3$

$\boxed{a = -3}$

$a + b = 2$

$-3 + b = 2$

$\boxed{b = 5}$

12. For what values of $a$ and $b$ will the function below be differentiable at $x = 1$?

$$f(x) = \begin{cases} 3ax^2 + 2bx + 1, & x \leq 1 \\ ax^4 - 4bx^2 - 3x, & x > 1 \end{cases}$$

① $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$ for $f(x)$ to be cont. at $x = 1$

$3a(1)^2 + 2b(1) + 1 = a(1)^4 - 4b(1)^2 - 3(1)$

$3a + 2b + 1 = a - 4b - 3$

$2a + 6b = -4$

② $\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^+} f'(x)$ for $f(x)$ to be diff. at $x=1$

$f'(x) = \begin{cases} 6ax + 2b, & x \leq 1 \\ 4ax^3 - 8bx - 3, & x > 1 \end{cases}$

$6a(1) + 2b = 4a(1)^3 - 8b(1) - 3$

$6a + 2b = 4a - 8b - 3$

$2a + 10b = -3$

$2a + 6b = -4$

$\ominus 2a \ominus 10b = \oplus 3$

$-4b = -1$

$\boxed{b = 1/4}$

$2a + 10(\frac{1}{4}) = -3$

$\boxed{a = -11/4}$