AP Calculus
Unit 3 – Rules of Differentiation

Day 3 Notes: Finding the Derivative of a Composite Function

Chain Rule of Differentiation of Composite Functions
If $h(x) = f(g(x))$, then $h'(x) = f'(g(x)) \cdot g'(x)$.

Find the derivative of each of the following functions by applying the chain rule.

$f(x) = (3x^2+2)^3$ $f'(x) = 3(3x^2+2)^2(6x)$ $= 18x(3x^2+2)^2$ $= 18x(9x^4+12x^2+4)$ $= \boxed{162x^5 + 216x^3 + 72x}$	$h(x) = \sqrt[3]{(x+2)^2}$ $h(x) = (x+2)^{2/3}$ $h'(x) = \frac{2}{3}(x+2)^{-1/3}(1)$ $= \boxed{\dfrac{2}{3\sqrt[3]{x+2}}}$
$F(x) = 5\sqrt[3]{x^2+2x}$ $F(x) = 5(x^2+2x)^{1/3}$ $F'(x) = \frac{5}{3}(x^2+2x)^{-2/3}(2x+2)$ $= \frac{5}{3} \cdot \frac{1}{\sqrt[3]{(x^2+2x)^2}} \cdot \frac{2x+2}{1}$ $= \boxed{\dfrac{10x+10}{3\sqrt[3]{(x^2+2x)^2}}}$	$h(x) = \sin^2(2x+1)$ $h(x) = [\sin(2x+1)]^2$ $h'(x) = 2[\sin(2x+1)]^1(\cos(2x+1))(2)$ $= \boxed{4\sin(2x+1)\cos(2x+1)}$

Now that you know "THE BIG THREE" rules of differentiation—product, quotient, and chain—let's see how the three can be incorporated with each other. Find the derivative of each of the following functions.

$f(x) = (5x)(\sqrt{x+3})$ $\quad f(x) = (5x)(x+3)^{1/2}$

Product Rule

chain rule

$f'(x) = (5)(x+3)^{1/2} + (5x)\frac{1}{2}(x+3)^{-1/2}(1)$

$= \frac{5\sqrt{x+3}\,(2\sqrt{x+3})}{(2\sqrt{x+3})} + \frac{5x}{2\sqrt{x+3}}$

$= \frac{10(x+3)+5x}{2\sqrt{x+3}} = \frac{15x+30}{2\sqrt{x+3}} = \boxed{\frac{15(x+2)}{2\sqrt{x+3}}}$

$g(x) = \sin\left(\frac{2x+1}{x-3}\right)$

Chain Rule

Quotient Rule

$g'(x) = \cos\left(\frac{2x+1}{x-3}\right) \cdot \left(\frac{(x-3)(2)-(2x+1)(1)}{(x-3)^2}\right)$

$\cos\left(\frac{2x+1}{x-3}\right) \cdot \left(\frac{\cancel{2x}-6-\cancel{2x}-1}{(x-3)^2}\right)$

$\boxed{\cos\left(\frac{2x+1}{x-3}\right)\left(\frac{-7}{(x-3)^2}\right)}$

$h(x) = \frac{\sqrt{2x+5}}{x-3}$ $\quad h(x) = \frac{(2x+5)^{1/2}}{(x-3)}$

Quotient Rule

Chain Rule

$h'(x) = \frac{(x-3)\cancel{\tfrac{1}{2}}(2x+5)^{-1/2}(\cancel{2}) - (2x+5)^{1/2}(1)}{(x-3)^2}$

$= \frac{(x-3)\frac{1}{\sqrt{2x+5}} - \sqrt{2x+5}}{(x-3)^2} \longrightarrow \frac{x-3}{\sqrt{2x+5}} - \frac{\sqrt{2x+5}\,(\sqrt{2x+5})}{(\sqrt{2x+5})}$

$\frac{x-3-(2x+5)}{\sqrt{2x+5}} = \frac{-x-8}{\sqrt{2x+5}}$

$\frac{-x-8}{\sqrt{2x+5}} \cdot \frac{1}{(x-3)^2} = \boxed{\frac{-x-8}{\sqrt{2x+5}\,(x-3)^2}}$