AP Calculus
Unit 3 – Rules of Differentiation

Day 2 Notes: Finding the Derivative of a Quotient of Two Functions

Example 1: Rewrite the function $f(x)=\dfrac{2x^3-3x^2+2}{x^2}$ as a function in polynomial form. Then, find $f'(x)$.

$f(x)=2x-3+2x^{-2}$

$f'(x)=2-4x^{-3}=\dfrac{(x^3)2}{(x^3)}-\dfrac{4}{x^3}=\boxed{\dfrac{2x^3-4}{x^3}}$

Quotient Rule of Differentiation

If $h(x)=\dfrac{f(x)}{g(x)}$, then $h'(x)=\dfrac{g(x)\cdot f'(x)-f(x)g'(x)}{[g(x)]^2}$

$h(x)=\dfrac{hi}{lo}$, then $h'(x)=\dfrac{lo\,d\,hi-hi\,d\,lo}{(lo)^2}$

To show that this rule works, let's apply this rule to the function $f(x)=\dfrac{2x^3-3x^2+2}{x^2}$ that we rewrote and differentiated as a polynomial-form above.

$f'(x)=\dfrac{(x^2)(6x^2-6x)-(2x^3-3x^2+2)(2x)}{(x^2)^2}=\dfrac{6x^4-\cancel{6x^3}+(-4x^4+\cancel{6x^3}-4x)}{x^4}=\dfrac{2x^4-4x}{x^4}$

$=\boxed{\dfrac{2x^3-4}{x^3}}$

Example 2: We will now use the quotient rule to derive the derivative formulas for the remaining trigonometric functions. Rewrite each function in terms of sine and/or cosine and differentiate using the Quotient Rule.

$f(\theta)=\tan\theta$	$f(\theta)=\cot\theta$
$f(\theta)=\dfrac{\sin\theta}{\cos\theta}$	$f(\theta)=\dfrac{\cos\theta}{\sin\theta}$
$f'(\theta)=\dfrac{(\cos\theta)(\cos\theta)-(\sin\theta)(-\sin\theta)}{(\cos\theta)^2}$	$f'(\theta)=\dfrac{(\sin\theta)(-\sin\theta)-(\cos\theta)(\cos\theta)}{(\sin\theta)^2}$
$=\dfrac{\cos^2\theta+\sin^2\theta}{(\cos^2\theta)}$	$=\dfrac{-\sin^2\theta-\cos^2\theta}{\sin^2\theta}$

$=\dfrac{1}{\cos^2\theta}=\boxed{\sec^2\theta}$

$=\dfrac{-1(\sin^2\theta+\cos^2\theta)}{\sin^2\theta}=\dfrac{-1}{\sin^2\theta}$

$=\boxed{-\csc^2\theta}$

$f(\theta) = \sec\theta$

$$f(\theta) = \frac{1}{\cos\theta}$$

$$f'(\theta) = \frac{\cancel{(\cos\theta)(0)} - (1)(-\sin\theta)}{(\cos\theta)^2}$$

$$= \frac{\sin\theta}{\cos^2\theta}$$

$$= \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\cos\theta}$$

$$= \boxed{\tan\theta\sec\theta}$$

$f(\theta) = \csc\theta$

$$f(\theta) = \frac{1}{\sin\theta}$$

$$f'(\theta) = \frac{\cancel{(\sin\theta)(0)} - (1)(\cos\theta)}{(\sin\theta)^2}$$

$$= -\frac{\cos\theta}{\sin^2\theta}$$

$$= -\frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta}$$

$$= \boxed{-\cot\theta\csc\theta}$$

Example 3: Find the derivative of each of the functions below by applying the quotient rule.

$$f(x) = \frac{x^2 - 2x}{x+2} \quad \cancel{x(x-2)}$$

$$f'(x) = \frac{(x+2)(2x-2) - (x^2-2x)(1)}{(x+2)^2}$$

$$= \frac{2x^2 - 2x + 4x - 4 - x^2 + 2x}{(x+2)^2}$$

$$= \boxed{\frac{x^2+4x-4}{(x+2)^2}}$$

$$g(x) = \frac{\tan x}{x+2}$$

$$g'(x) = \frac{(x+2)(\sec^2 x) - (\tan x)(1)}{(x+2)^2}$$

$$= \boxed{\frac{x\sec^2 x + 2\sec^2 x - \tan x}{(x+2)^2}}$$

$$h(\theta) = \frac{\sin\theta}{1-\cos\theta}$$

$$h'(\theta) = \frac{(1-\cos\theta)(\cos\theta) - (\sin\theta)(\sin\theta)}{(1-\cos\theta)^2}$$

$$= \frac{\cos\theta - \cos^2\theta - \sin^2\theta}{(1-\cos\theta)^2}$$

$$= \frac{-1+\cos\theta}{(1-\cos\theta)^2}$$

$$= \frac{-(1-\cos\theta)}{(1-\cos\theta)^2} = \boxed{\frac{-1}{1-\cos\theta}}$$

$$f(x) = \frac{3 - \frac{1}{x}}{x+5} \quad \frac{(x)}{(x)}$$

$$f(x) = \frac{3x-1}{x^2+5x}$$

$$f'(x) = \frac{(x^2+5x)(3) - (3x-1)(2x+5)}{(x^2+5x)^2}$$

$$= \frac{3x^2+15x - (6x^2+15x-2x-5)}{[x(x+5)]^2}$$

$$= \frac{-3x^2+2x+5}{x^2(x+5)^2}$$

$$= \frac{-(3x^2-2x-5)}{x^2(x+5)^2} = \boxed{\frac{-(3x-5)(x+1)}{x^2(x+5)^2}}$$

Show, using the quotient rule, that if $f(x) = \dfrac{x^2+3x+2}{x^2-1}$, then $f'(x) = -\dfrac{3}{(x-1)^2}$.

$$f'(x) = \frac{(x^2-1)(2x+3) - (x^2+3x+2)(2x)}{(x^2-1)^2}$$

$$= \frac{\cancel{2x^3}+3x^2-2x-3-\cancel{2x^3}-6x^2-4x}{[(x+1)(x-1)]^2}$$

$$= \frac{-3x^2-6x-3}{(x+1)^2(x-1)^2} = \frac{-3(x^2+2x+1)}{(x+1)^2(x-1)^2} = \frac{-3\cancel{(x+1)(x+1)}}{\cancel{(x+1)^2}(x-1)^2}$$

$$= \boxed{\frac{-3}{(x-1)^2}}$$

Similar to the Product Rule, there is a very valuable lesson that we must learn when we are introduced to the quotient rule. In the box below, first factor and simplify the function, $f(x) = \dfrac{x^2+3x+2}{x^2-1}$, from above. Then, differentiate using the quotient rule

$$f(x) = \frac{(x+2)\cancel{(x+1)}}{\cancel{(x+1)}(x-1)} \qquad f(x) = \frac{(x+2)}{(x-1)}$$

$$f'(x) = \frac{(x-1)(1)-(x+2)(1)}{(x-1)^2} = \frac{x-1-x-2}{(x-1)^2} = \boxed{\frac{-3}{(x-1)^2}}$$

What is the lesson to be learned from the algebraic analysis above?

If the rational function can be factored and simplified before differentiate, Then DO IT!

GROUPS:

Let $f(x)$ and $g(x)$ be differentiable functions such that the following values are true.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	2	–1	9	–1
3	–5	–3	–4	6
4	1	7	8	–2

Estimate the value of $g'(2.5)$.

$(2,-1) \quad (3,-3)$

$$m = \frac{-3-(-1)}{3-2} = \frac{-2}{1} \approx \boxed{-2}$$

If $p(x) = \dfrac{g(x)}{f(x)}$, what is the value of $p'(4)$? What does this value say about the graph of $p(x)$ when $x = 4$? Give a reason for your answer.

$$p'(x) = \frac{f(x)g'(x) - g(x)f'(x)}{[f(x)]^2}$$

$$p'(4) = \frac{f(4)g'(4) - g(4)f'(4)}{[f(4)]^2}$$

$$= \frac{(1)(-2)-(7)(8)}{(1)^2}$$

$$= \frac{-2-56}{1} = -58$$

Since $p'(4)<0$, the graph of $p(x)$ is decreasing.

If $q(x) = (2x^2)\left(\dfrac{f(x)}{g(x)}\right)$, what is the value of $q'(2)$?

$$q'(x) = (4x)\left(\frac{f(x)}{g(x)}\right) + (2x^2)\left(\frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}\right)$$

$$q'(2) = (4(2))\left(\frac{f(2)}{g(2)}\right) + (2(2)^2)\left(\frac{g(2)f'(2) - f(2)g'(2)}{(g(2))^2}\right)$$

$$(8)\left(\frac{2}{-1}\right) + (8)\left(\frac{(-1)(9)-(2)(-1)}{(-1)^2}\right) = -16 + 8(-7) = -16-56 = \boxed{-72}$$

Find the equation of the line tangent to the graph of $v(x) = \dfrac{3x}{g(x)}$ when $x = 3$.

P.O.T $\rightarrow$ $v(3) = \dfrac{3(3)}{g(3)} = \dfrac{9}{-3} = -3$ $\quad (3,-3)$

S.O.T $\rightarrow$ $v'(x) = \dfrac{g(x)(3) - (3x)g'(x)}{(g(x))^2}$ $\quad \boxed{y+3 = -7(x-3)}$

$$v'(3) = \frac{g(3)(3) - (3(3))g'(3)}{[g(3)]^2} = \frac{(-3)(3)-(9)(6)}{(-3)^2}$$

$$= \frac{-9-54}{9} = -7$$