AP Calculus
Unit 3 – Rules of Differentiation

Day 1 Notes: Finding the Derivative of a Product of Two Functions

Example 1: Rewrite the function $f(x) = (2x-3)(x^2-2x+1)$ as a cubic function. Then, find $f'(x)$. What does this equation of $f'(x)$ represent, again?

$f(x) = 2x^3 - 4x^2 + 2x - 3x^2 + 6x - 3$

$f(x) = 2x^3 - 7x^2 + 8x - 3$

$f'(x) = 6x^2 - 14x + 8$ ← Eqn. that can be used to find the slope of any tangent line drawn to $f(x)$.

Product Rule of Differentiation

If $h(x) = f(x) \cdot g(x)$, then $h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.

To show that this rule works, let's apply this rule to the function $f(x) = (2x-3)(x^2-2x+1)$ that we rewrote and differentiated as a polynomial above.

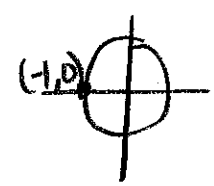
$f'(x) = (2)(x^2-2x+1) + (2x-3)(2x-2)$

$= 2x^2 - 4x + 2 + 4x^2 - 4x - 6x + 6$

$f'(x) = 6x^2 - 14x + 8$ ✓

Students often wonder why this rule is so important if we could just rewrite as a polynomial and easily differentiate it. The answer to that question is simple. If it is possible to rewrite as a polynomial, always do so. But in the case of the function $g(x) = x^2 \sin x$, *there is no way to rewrite as a polynomial.*

Example 2: Apply the product rule to find the slope of the normal line to the graph of $g(x) = (x^2)(\sin x)$ when $x = \pi$.



$g'(x) = (2x)(\sin x) + (x^2)(\cos x)$

$g'(\pi) = (2\pi)(\sin \pi) + (\pi^2)(\cos \pi)$

$= 2\pi(0) + (\pi^2)(-1)$

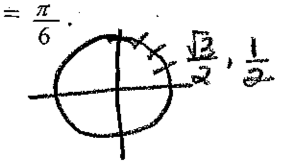
$= -\pi^2$ ← slope of tangent

$\boxed{\text{Slope of normal} = 1/\pi^2}$

Example 3: Use the product rule to find the derivative of each of the following functions.

$f(x)=\left(2x^2+3x\right)\left(x^2-3\right)$ $f'(x)=(4x+3)(x^2-3)+(2x^2+3x)(2x)$ $=4x^3-12x+3x^2-9+4x^3+6x^2$ $\boxed{f'(x)=8x^3+9x^2-12x-9}$	$g(x)=\sqrt{x}\left(x^2-3x+2\right)$ $g(x)=x^{1/2}(x^2-3x+2)$ $g'(x)=\frac{1}{2}x^{-1/2}(x^2-3x+2)+(x^{1/2})(2x-3)$ $\frac{1}{2}x^{3/2}-\frac{3}{2}x^{1/2}+x^{-1/2}+2x^{3/2}-3x^{1/2}$ $\frac{5}{2}x^{\frac{3}{2}}-\frac{9}{2}x^{1/2}+x^{-1/2}$ $\frac{(\sqrt{x})}{(\sqrt{x})}\frac{5\sqrt{x^3}}{2}-\frac{(\sqrt{x})9\sqrt{x}}{(\sqrt{x})2}+\frac{1(2)}{\sqrt{x}(2)}=\boxed{\frac{5x^2-9x+2}{2\sqrt{x}}}$
$f(x)=x^3\sin x$ $f'(x)=(3x^2)(\sin x)+(x^3)(\cos x)$ $f'(x)=3x^2\sin x+x^3\cos x$ $\boxed{f'(x)=x^2(3\sin x+x\cos x)}$	$h(x)=(3x+2)\cos x$ $h'(x)=(3)(\cos x)+(3x+2)(-\sin x)$ $\boxed{h'(x)=3\cos x-3x\sin x-2\sin x}$
$g(x)=3\theta+(\theta)(\sin\theta)$ $g'(x)=3+(1)(\sin\theta)+(\theta)(\cos\theta)$ $\boxed{g'(x)=3+\sin\theta+\theta\cos\theta}$	$h(x)=(\sin x)(\cos x)$ $h'(x)=(\cos x)(\cos x)+(\sin x)(-\sin x)$ $=\cos^2x-\sin^2x \rightarrow \sin^2x=1-\cos^2x$ $=\cos^2x-(1-\cos^2x)$ $=\cos^2x-1+\cos^2x$ $\boxed{h'(x)=2\cos^2x-1}$

Example 4: Find the equation of the line tangent to the graph of $g(t)=(t^2)(\cos t)$ when $t=\frac{\pi}{6}$.



$*$P.O.T. $\rightarrow g\left(\frac{\pi}{6}\right)=\left(\frac{\pi}{6}\right)^2\cos\left(\frac{\pi}{6}\right)$ $\left(\frac{\pi}{6},\frac{\pi^2\sqrt{3}}{72}\right)$

$\frac{\pi^2}{36}\left(\frac{\sqrt{3}}{2}\right)$

$*$S.O.T $\rightarrow g'(t)=(2t)(\cos t)+(t^2)(-\sin t)$

$g'\left(\frac{\pi}{6}\right)=2\left(\frac{\pi}{6}\right)\left(\cos\frac{\pi}{6}\right)+\left(\frac{\pi}{6}\right)^2\left(-\sin\left(\frac{\pi}{6}\right)\right)$

$=\left(\frac{\pi}{3}\right)\left(\frac{\sqrt{3}}{2}\right)+-\frac{\pi^2}{36}\left(\frac{1}{2}\right)$

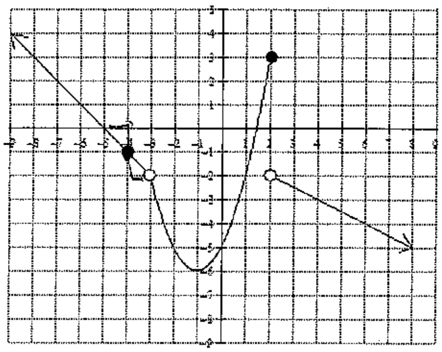
$=\frac{(12)\pi\sqrt{3}}{(12)6}-\frac{\pi^2}{72}$

$=\frac{12\pi\sqrt{3}-\pi^2}{72}$

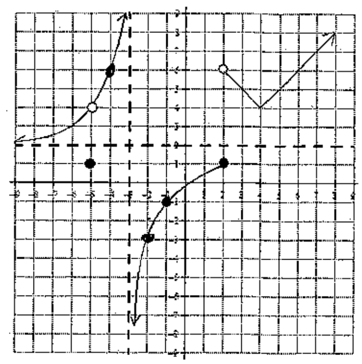
$$\boxed{y-\frac{\pi^2\sqrt{3}}{72}=\frac{12\pi\sqrt{3}-\pi^2}{72}\left(x-\frac{\pi}{6}\right)}$$

Example 5: Below are graphs of two functions—$f(x)$ and $g(x)$. Let $P(x) = f(x) \cdot g(x)$ and let $R(x) = x^2 \cdot g(x)$. Use the graphs to answer the questions that follow.

Graph of $f(x)$



Graph of $g(x)$



If $g'(-4) = 2$, what is the value of $P'(-4)$?	If $R'(-2) = 20$, what is the value of $g'(-2)$?
$P(x) = f(x) \cdot g(x)$ $P'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$ $P'(-4) = \underbrace{f'(-4)} \cdot g(-4) + f(-4) \cdot g'(-4)$ slope at $x=-4$ $P'(-4) = (-1)(6) + (-1) \cdot (2)$ $= -6 - 2$ $= \boxed{-8}$	$R(x) = x^2 \cdot g(x)$ $R'(x) = (2x) \cdot g(x) + (x^2) g'(x)$ $\underbrace{R'(-2)} = (2 \cdot -2) g(-2) + (-2)^2 \underline{g'(-2)}$ $20 = -4(-3) + 4(x)$ $20 = 12 + 4x$ $8 = 4x$ $x = 2$ $\boxed{g'(-2) = 2}$
Find the equation of the line tangent to the graph of $P(x)$ when $x = -4$.	Find the equation of the line tangent to the graph of $R(x)$ when $x = -2$.
P.O.T → $P(-4) = f(-4) \cdot g(-4)$ $= (-1)(6) = -6$ $(-4, -6)$ S.O.T → $P'(-4) = -8$ (from above) $\boxed{y + 6 = -8(x + 4)}$	P.O.T → $R(-2) = (-2)^2 \cdot g(-2)$ $= 4(-3) = -12$ $(-2, -12)$ S.O.T → $R'(-2) = 20$ [from above) $\boxed{y + 12 = 20(x + 2)}$

Example 6: Let $f(x)$ and $g(x)$ be differentiable functions such that the following values are true.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
4	1	7	2	–3
3	–2	–3	–4	2
–1	2	–2	1	–1

Estimate the value of $f'(3.5)$.

$(4,1)$ & $(3,-2)$

$\text{slope} = \frac{-2-1}{3-4} = \frac{-3}{-1} = 3$

$\boxed{f'(3.5) \approx 3}$

If $q(x) = 2f(x) - 4g(x)$, what is the value of $q'(4)$?

$q(x) = 2f(x) - 4g(x)$

$q'(x) = 2f'(x) - 4g'(x)$

$q'(4) = 2f'(4) - 4g'(4)$

$q'(4) = 2(2) - 4(-3)$

$= 4 + 12 = \boxed{16}$

If $p(x) = -2f(x)g(x)$, what is the value of $p'(3)$?

$p'(x) = -2f'(x)g(x) + -2f(x)g'(x)$

$p'(3) = -2f'(3)g(3) - 2f(3)g'(3)$

$= -2(-4)(-3) - 2(-2)(2)$

$= -24 + 8 = \boxed{-16}$

Find the equation of the line tangent to the graph of $v(x) = x^3 \cdot f(x)$ when $x = -1$.

P.O.T $\rightarrow$ $v(-1) = (-1)^3 \cdot f(-1)$

$(-1)(2) = -2$

$(-1,-2)$

S.O.T $\rightarrow$ $v'(x) = (3x^2)\underbrace{f(x)}_{f(-1)} + x^3 \underbrace{f'(x)}_{f'(-1)}$

$= 3(-1)^2(2) + (-1)^3(1)$

$= 6 - 1 = 5$

$\boxed{y + 2 = 5(x+1)}$

If $k(x) = (2f(x) + 3)(3 - g(x))$, what is the value of $k'(3)$?

$k'(x) = (2f'(x))(3 - g(x)) + (2f(x) + 3)(-g'(x))$

$k'(3) = (2f'(3))(3 - g(3)) + (2f(3) + 3)(-g'(3))$

$= (2(-4))(3 - (-3)) + (2(-2) + 3)(-2)$

$= (-8)(6) + (-1)(-2)$

$= -48 + 2 = \boxed{-46}$