AP Calculus AB **Name:** Answer Key*

Unit 2 – Conceptualizing the Derivative - REVIEW

1. If $g'(x) = (x-3)^2(x+1)$, determine on what intervals the graph of $g(x)$ is increasing or decreasing and identify the value(s) of x at which $g(x)$ has a relative maximum or minimum. Justify your reasoning and show your work.

$(x-3)^2(x+1) = 0$

$x = 3,\ x = -1$

	-2		0		4
$g'(x)$	$\ominus$	-1	$\oplus$	3	$\oplus$
	$(+)(-)$		$(+)(+)$		$(+)(+)$

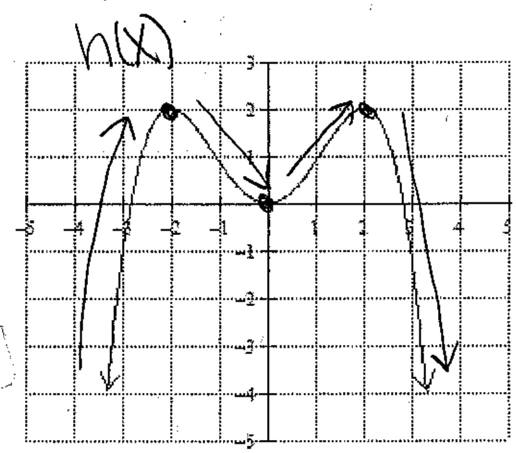
increasing → $(-1,3) \cup (3,\infty)$ b/c $g'(x) > 0$

decreasing → $(-\infty,-1)$ b/c $g'(x) < 0$

rel. min at $x=-1$ b/c $g'(x) = 0$ & $g'(x)$ goes from negative to positive

NO rel. max b/c $g'(x)$ never goes from positive to negative.

For exercises 2 – 4, use the graph of the function, $h(x)$, pictured to the right. Use the graph to identify the following. **Provide written justification.**



~~2. On what interval(s) is $h'(x) < 0$?~~

If $h'(x)$ is negative, then $h(x)$ is Decreasing.

$(-2,0) \cup (2,\infty)$

3. On what interval(s) is $h'(x) > 0$?

If $h'(x)$ is positive, then $h(x)$ is increasing.

$(-\infty,-2) \cup (0,2)$

4. At what value(s) of x does $h'(x)$ change from positive to negative? From negative to positive?

If $h'(x)$ changes from positive to negative then $h(x)$ has a relative max.

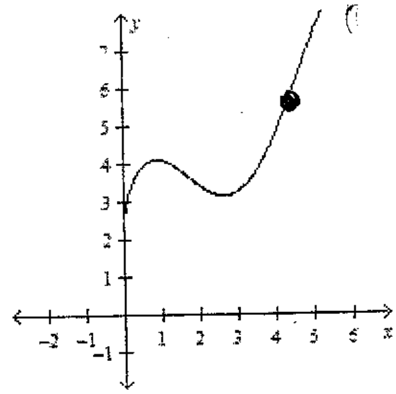
$x=-2$ & $x=2$

If $h'(x)$ changes from negative to positive then $h(x)$ has a relative min.

$x=0$

#'s 5 – 10: CALCULATOR PERMITTED

Consider the function, $g(x) = 3\sqrt{x} + 2\cos x$, which is pictured to the right.
Graph of $g(x)$ on the interval $0 < x \le 2\pi$ and g(x) is concave up at x = 4.5.



5. Algebraically find $g'(x)$. Express your answer as a single rational function with positive exponents.

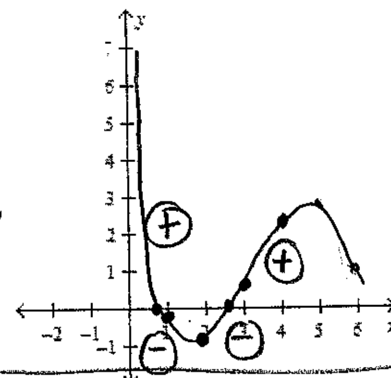
$g(x) = 3x^{1/2} + 2\cos x$

$g'(x) = \frac{3}{2}x^{-1/2} + 2(-\sin x)$

$\frac{3}{2\sqrt{x}} - \frac{2\sin x\,(2\sqrt{x})}{1\,(2\sqrt{x})} = \boxed{\frac{3 - 4\sqrt{x}\sin x}{2\sqrt{x}}}$

6. Sketch the graph of $g'(x)$ on the axes to the right. Then, state the value(s) of x where the graph of $g(x)$ has a relative maximum and/or minimum. Justify your answers.

(use calc)



x	Y
0	undef.
1	-.183
2	-.758
3	0.584
4	2.264
5	2.589
6	1.171

$y_1 = \frac{3 - 4\sqrt{x}\sin x}{2\sqrt{x}}$ $\quad y_2 = 0$ } 2nd Trace Intersect

x = 0.907 is rel. max b/c g'(x) goes from positive to negative

X = 2.664 is a rel. min b/c g'(x) goes from negative to positive.

7. Based on the graph of $g(x)$, will the slope of the tangent line be positive or negative at $x = 4.5$. Give a reason for your answer.

Slope of tangent will be positive because g(x) is increasing at x = 4.5, which means g'(x) > 0.

8. Use the equation of $g'(x)$ to verify your answer to exercise 7.

$g'(4.5) = \frac{3 - 4\sqrt{4.5}\,(\sin(4.5))}{2(\sqrt{4.5})} = \boxed{2.662 > 0}$ ✓

9. Find the equation of the tangent line to the graph of $g(x)$ when $x = 4.5$.

$g(4.5) = 5.942$ point → (4.5, 5.942)

$g'(4.5) = 2.662$ ← slope

$\boxed{y - 5.942 = 2.662(x - 4.5)}$

10. Use the equation of the tangent line at $x = 4.5$ to approximate $g(4.1)$. Then, use the equation of g to find $g(4.1)$. Is the tangent line approximation an under or over approximation? Explain why this is true based on the graph.

Plug in x = 4.1 to $y - 5.942 = 2.662(x - 4.5)$

$y - 5.942 = 2.662(4.1 - 4.5)$

$y = \boxed{4.8772}$

$g(4.1) = 3\sqrt{4.1} + 2\cos(4.1)$

$g(4.1) = \boxed{4.925}$

Since g(x) is concave up, this is under approx.

$$h'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

11. For what function does $\lim_{h \to 0} \frac{2\sin(x+h) - 2\sin x}{h}$ give the derivative? Find the limit.

$\boxed{f(x) = 2\sin x} \longrightarrow \boxed{f'(x) = 2\cos x}$

12. Find $\lim_{h \to 0} \frac{(x+h)^5 - x^5}{h}$.

$f(x) = x^5$

$\boxed{f'(x) = 5x^4}$

13. Find $\lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h}$.

$\lim_{h \to 0} \frac{-(\sqrt{x+h} - \sqrt{x})}{h}$

$-\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$ } means $f(x) = \sqrt{x}$

$= x^{1/2}$

$f'(x) = \frac{-1}{2} x^{-1/2}$

$\boxed{f'(x) = \frac{-1}{2\sqrt{x}}}$

14. If $f(x) = \frac{3x}{\sqrt{x}}$, what is the slope of the normal line to the graph of $f(x)$ when $x = 4$?

slope of tangent $\rightarrow$ find $f'(x)$. $f(x) = \frac{3x}{x^{1/2}}$

$f(x) = 3x^{1/2} \rightarrow f'(x) = \frac{3}{2} x^{-1/2}$

$f'(4) = \frac{3}{2}(4)^{-1/2}$ or $\frac{3}{2\sqrt{4}} = \frac{3}{4}$

slope of tangent

since slope of tangent is 3/4, the slope of the normal is $\boxed{-4/3}$

15. If $2x - 3 = 5(y + 1)$ is the equation of the normal line to the graph of $f(x)$ when $x = a$, find the value of $f'(a)$. Show your work and explain your reasoning.

$2x - 3 = 5(y+1)$

$2x - 3 = 5y + 5$

$2x - 8 = 5y$

$\frac{2}{5}x - \frac{8}{5} = y$

slope of normal

$f'(a)$ is the slope of the tangent at $x = a$.

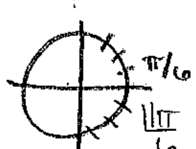
$\boxed{-5/2}$ $\leftarrow$ slope of tangent

16. On the interval $[0, 2\pi)$, find the ~~coordinates~~ x-value of the relative minimum(s) of $f(\theta) = \sqrt{3}\theta - 2\sin\theta$.

$f'(\theta) = \sqrt{3} - 2\cos\theta$

$\sqrt{3} - 2\cos\theta = 0$

$\cos\theta = \frac{\sqrt{3}}{2}$



$\theta = \pi/6, \frac{11\pi}{6}$

π/12 π

0 ⊖ $\frac{\pi}{6}$ ⊕ $\frac{11\pi}{6}$ ⊖ 2π

$\sqrt{3} - 2\cos(\frac{\pi}{12})$ $\sqrt{3} - 2\cos(\pi)$ $\sqrt{3} - 2(+)$

$\sqrt{3} - 2(+)$

$\boxed{\text{rel. min at } x = \pi/6}$

The derivative of a function $f(x)$ is $f'(x) = (3-x)^2(x+5)$. Use this derivative for exercises 17 and 18.

17. At what value(s) of x does the graph of $f(x)$ have a relative maximum? Justify your answer.

$(3-x)^2(x+5)=0$

$3-x=0$

$x=3 \quad x=-5$

-7, -5, 0, 3, 4

$-$, $+$, $+$

$(+)(-)$, $(+)(+)$, $(+)(+)$

f(x) does not have a relative maximum b/c f'(x) doesn't change from positive to negative.

18. Use the equation of the tangent line to approximate the value of $f(2.1)$ if $f(2) = -3$. If it is known that $f(x)$ is concave down at $x = 2$, is this approximation an over or under approximation of $f(2.1)$? Give a reason for your answer.

Tangent line at $x=2$:

point $(2,-3)$

slope: $f'(2) = (3-2)^2(2+5)$

$= (1)(7)$

$= 7$

$y+3 = 7(x-2)$

plug in $x=2.1$

$y+3 = 7(2.1-2)$

$y+3 = 7(.1)$

$y+3 = .7$

$\boxed{y=-2.3}$

Since f(x) is concave down at x=2, this is an over approximation of f(2.1)

Free Response Practice
Calculator Permitted

Pictured to the right is the graph of the first derivative, $f'(x)$, of a polynomial function $f(x)$, such that $f(-1) = 2$. Additionally, the graph of $f(x)$ is concave down when $x = -1$.

a. Approximate the value of $f(-0.9)$ using the equation of the tangent line drawn to the graph of $f(x)$ when $x = -1$. Is this approximation greater or less than the actual value of $f(-0.9)$? Give a reason for your answer.

Tangent Line at $X=-1$:

- point → $(-1, 2)$
- slope → $f'(-1) = -3$

look at graph

$\boxed{y-2=-3(x+1)}$

plug in $x=-.9$ → $y-2=-3(-.9+1)$

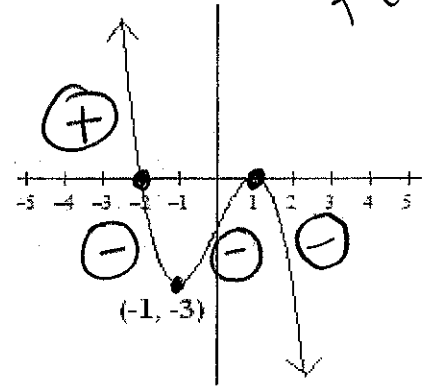
$y-2=-.3$

$\boxed{y=1.7}$

f(-.9) is an over approx. Since the graph of f(x) is concave down at $x=-1$.

Graph of f '(x)

f'(x)



b. On what interval(s) is the graph of $f(x)$ increasing or decreasing? Give a reason for your answer.

increasing → $(-\infty, -2)$ b/c $f'(x) > 0$

decreasing → $(-2, -1) \cup (1, \infty)$ b/c $f'(x) < 0$

c. At what value(s) of x does the graph of $f(x)$ have a relative maximum? Justify your answer.

relative maximum at $x=-2$ b/c $f'(x)=0$ and $f'(x)$ goes from positive to negative.

d. At what value(s) of x does the graph of $f(x)$ have a relative minimum? Justify your answer.

No relative minimum b/c $f'(x)$ never goes from negative to positive.