AP Calculus
Unit 2 – Conceptualizing the Derivative

Day 3 Notes: Analytically Finding the Derivative of Polynomial, Polynomial Type, Sine, and Cosine Functions

***Derivative of a Constant:**
If $f(x) = c$, where c is any constant, then $f'(x) =$ 0.

***Power Rule for Differentiation:**
"take the derivative" $\rightarrow \frac{d}{dx}\left[x^n\right] = n \cdot x^{n-1}$

(In order to apply the Power Rule for Differentiation, the equation must be written in "polynomial form.")

Example 1: Find the derivative of each function.

Function, $f(x)$	**Derivative, $f'(x)$**
$f(x) = 3x^2 - 2x + 3$	$f'(x) = 6x - 2$
$f(x) = -5x^3 + 2x^2 - 3x + 1$	$f'(x) = -15x^2 + 4x - 3$
$f(x) = 6 - 3x^3 + 6x^4$	$f'(x) = -9x^2 + 24x^3$
$f(x) = -2x^{-1} + 3x^{-2}$	$f'(x) = 2x^{-2} - 6x^{-3}$
$f(x) = 6x^{\frac{2}{3}} + 4x^{-2}$	$f'(x) = 4x^{-1/3} - 8x^{-3}$
$f(x) = -6x^{-\frac{1}{2}} + 3x^{\frac{1}{2}}$	$f'(x) = 3x^{-3/2} + \frac{3}{2}x^{-1/2}$

$a^0 = 1 \quad a^{-n} = \left(\frac{1}{a}\right)^n \quad a^{m/n} = \sqrt[n]{a^m}$

Example 2: Find $f'(x)$ for each of the following functions. Leave your answers with no negative or rational exponents and as single rational functions, when applicable.

$$f(x) = \frac{2}{x^2} - 4x^3$$

$f(x) = 2x^{-2} - 4x^3$

$f'(x) = -4x^{-3} - 12x^2$

$\frac{-4}{x^3} - \frac{12x^2}{1}\frac{(x^3)}{(x^3)}$

$\boxed{f'(x) = \frac{-4 - 12x^5}{x^3}}$

$$f(x) = \frac{3x^4 - 3x^2 - 2x}{x}$$

$f(x) = 3x^3 - 3x - 2$

$\boxed{f'(x) = 9x^2 - 3}$

$$f(x) = (x+3)(x+2)(2x+1)$$

$(x^2 + 5x + 6)(2x+1)$

$2x^3 + x^2 + 10x^2 + 5x + 12x + 6$

$f(x) = 2x^3 + 11x^2 + 17x + 6$

$\boxed{f'(x) = 6x^2 + 22x + 17}$

$$f(x) = \frac{x^3 - 5x^2}{x^5}$$

$x^{-5}(x^3 - 5x^2)$

$f(x) = x^{-2} - 5x^{-3}$

$f'(x) = -2x^{-3} + 15x^{-4}$

$\frac{-2(x)}{x^3(x)} + \frac{15}{x^4} = \boxed{\frac{-2x+15}{x^4} = f'(x)}$

$$f(x) = \frac{3x}{\sqrt[3]{x^2}}$$

$\frac{3x}{x^{2/3}}$ $\quad f(x) = x^{-2/3} \cdot 3x$

$f(x) = 3x^{1/3}$

$f'(x) = x^{-2/3}$

$\frac{1}{x^{2/3}}$ $\quad \boxed{f'(x) = \frac{1}{\sqrt[3]{x^2}}}$

$$f(x) = -4x^{\frac{3}{4}} + 2x^{\frac{1}{4}}$$

$f'(x) = -3x^{-1/4} + \frac{1}{2}x^{-3/4}$

$\frac{-3\,(2)(x^{2/4})}{x^{1/4}(2)(x^{2/4})} + \frac{1}{2x^{3/4}}$

$\frac{-6x^{\frac{1}{2}} + 1}{2x^{3/4}}$

$\boxed{f'(x) = \frac{-6\sqrt{x} + 1}{2\sqrt[4]{x^3}}}$

$\lim_{h\to 0} \frac{1-\cosh}{h} = 0$ $\lim_{h\to 0} \frac{\sinh}{h} = 1$

Finding the Derivative of Sine & Cosine using the Definition of Derivative:

Remember two trigonometric identities that we will use to find the derivatives of the sine and cosine functions.

$\cos(a+b) =$ cosacosb − sinasinb

$\sin(a+b) =$ sinacosb + cosasinb

Use $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ to find $f'(x)$ for each of the following functions. Your results will show the derivative of the sine and cosine functions.

$f(x) = \sin x$	$f(x) = \cos x$
$\lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$	$\lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h}$
$\frac{\sin x\cosh + \cos x\sinh - \sin x}{h}$	$\frac{\cos x\cosh - \sin x\sinh - \cos x}{h}$
$\frac{\sin x\cosh - \sin x}{h} + \frac{\cos x\sinh}{h}$	$\frac{\cos x\cosh - \cos x}{h} + \frac{-\sin x\sinh}{h}$
$\sin x\frac{(\cosh - 1)}{h} + \cos x\frac{\sinh}{h}$	$\cos x\frac{(\cosh - 1)}{h} + -\sin x\frac{\sinh}{h}$
$\lim_{h\to 0} (\sin x)\left(\frac{\cosh-1}{h}\right) + \lim_{h\to 0} (\cos x)\left(\frac{\sinh}{h}\right)$	$\lim_{h\to 0} (\cos x)\left(\frac{\cosh-1}{h}\right) + \lim_{h\to 0} (-\sin x)\left(\frac{\sinh}{h}\right)$
$\sin x \cdot (0)$	$\cancel{\cos x(0)} + (-\sin x)(1)$
$0 + \cos x(1)$	
$f'(x) = \cos x$	$f'(x) = -\sin x$

***Derivative of Sine & Cosine:**

$\frac{d}{dx}[\sin x] =$ cosx $\frac{d}{dx}[\cos x] =$ −sinx

Example 3: For each of the following functions, find the equation of the tangent line to the graph of the function at the given point.

$g(\theta) = 2\theta + 3\cos\theta$ when $\theta = \pi$

P.O.T $\rightarrow$ $g(\pi) = 2\pi + 3\cos(\pi)$
$= 2\pi + 3(-1)$
$= 2\pi - 3$

S.O.T $\rightarrow$ $g'(\theta) = 2 + 3(-\sin\theta)$
$= 2 - 3\sin\theta$

$g'(\pi) = 2 - 3\sin(\pi)$
$= 2$

$y - (2\pi - 3) = 2(x - \pi)$
$y - \cancel{2\pi} + 3 = 2x - \cancel{2\pi}$
$\boxed{y = 2x - 3}$

$f(\theta) = 4\sin\theta - \theta$ when $\theta = \frac{\pi}{2}$

P.O.T $\rightarrow$ $f(\frac{\pi}{2}) = 4\underbrace{\sin(\tfrac{\pi}{2})} - \frac{\pi}{2}$
$= 4(1) - \frac{\pi}{2}$
$= 4 - \frac{\pi}{2}$

S.O.T $\rightarrow$ $f'(\theta) = 4(\cos\theta) - 1$
$f'(\frac{\pi}{2}) = 4\underbrace{\cos(\tfrac{\pi}{2})} - 1$
$4(0) - 1 = -1$

$y - (4 - \frac{\pi}{2}) = -1(x - \pi/2)$
$y - 4 + \cancel{\frac{\pi}{2}} = -x + \cancel{\frac{\pi}{2}}$
$\boxed{y = -x + 4}$

Horizontal Tangent:
Horizontal Tangent occurs when $\boxed{f'(x) = 0}$ (slope of tangent = 0).

"possible rel. max or min" Zero Slope

Example 4: At what value(s) of x will the function $f(x) = x^3 + x$ have a horizontal tangent?

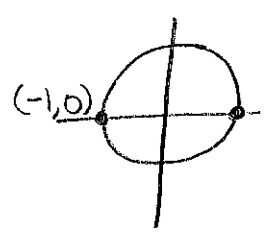
occurs when $f'(x) = 0$

① Find $f'(x)$. $f'(x) = 3x^2 + 1$

② Set $f'(x) = 0$ $3x^2 + 1 = 0$
$3x^2 = -1$
$x^2 = -1/3$
$x = \pm\sqrt{-1/3}$ = undefined $\boxed{\text{No Solution}}$

Example 5: At what value(s) of θ at which the function $f(\theta) = \theta + \sin\theta$ has a horizontal tangent on the interval $[0, 2\pi)$?

$f'(\theta) = 1 + \cos\theta$



$1 + \cos\theta = 0$
$\cos\theta = -1$
$\theta = \cos^{-1}(-1)$
$\boxed{\theta = \pi}$