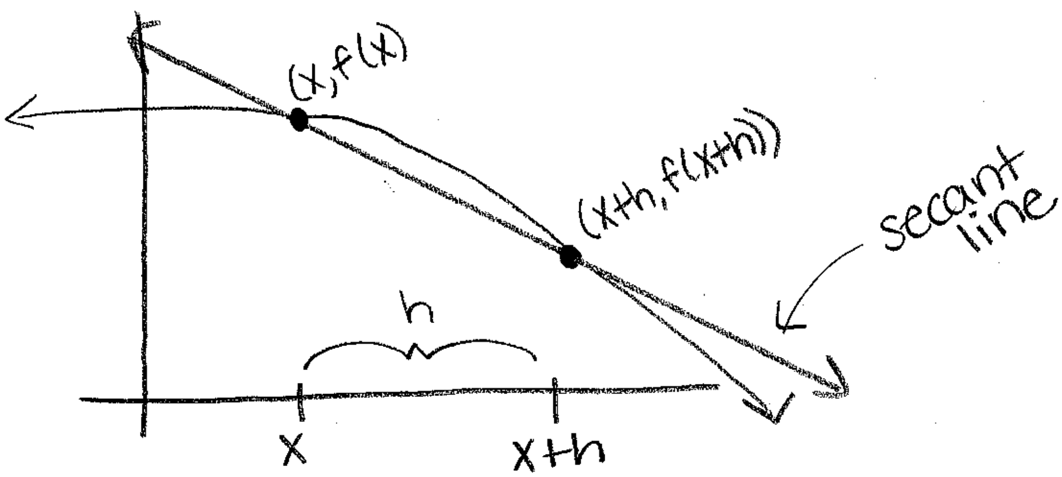
AP Calculus
Unit 2 – Conceptualizing the Derivative

Day 1 Notes: The Difference Quotient (A First Look at the Derivative)

Today we are introduced to the concept with which we will spend our greatest amount of time investigating in Calculus AB—the derivative.



What does the expression $\frac{f(x+h)-f(x)}{(x+h)-x}$ represent? What does this expression simplify to?

represents the SLOPE of the secant line

$\frac{f(x+h)-f(x)}{h}$

As h, the distance between the x – values, x and $(x+h)$, approaches zero, what happens to the secant line?

As $h \to 0$, the secant line eventually becomes a TANGENT LINE.

What does the limit $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ represent? ← DIFFERENCE QUOTIENT

The SLOPE of the TANGENT LINE when $(x, f(x))$ & $(x+h, f(x+h))$ become the same point.

Definition of the Derivative and What It Represents Graphically

* The DERIVATIVE $f'(x)$, is a function that results from $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ and that can be used to find the SLOPE of any tangent line drawn to the graph of $f(x)$. *

A.K.A. Find the derivative.

Example 1: Suppose $f(x) = -x^2 - 4x + 1$. Find $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$.

$$\lim_{h\to 0} \frac{-(x+h)^2 - 4(x+h) + 1 - (-x^2 - 4x + 1)}{h}$$

$-(x^2 + 2xh + h^2) - 4x - 4h + 1$

$\cancel{-x^2} - 2xh - h^2 \cancel{-4x} - 4h \cancel{+1}$

$\cancel{+x^2} \cancel{+4x} \cancel{-1}$

$$\lim_{h\to 0} \frac{-2xh - h^2 - 4h}{h} = \lim_{h\to 0} \frac{\cancel{h}(-2x - h - 4)}{\cancel{h}}$$

$$\lim_{h\to 0} (-2x - h - 4) = -2x - 0 - 4 = -2x - 4$$

$$\boxed{f'(x) = -2x - 4}$$

Find the equation of the tangent line to $f(x)$ at each of the points below. Then, draw the graphs of the tangent lines on the grid above where $f(x)$ is graphed.

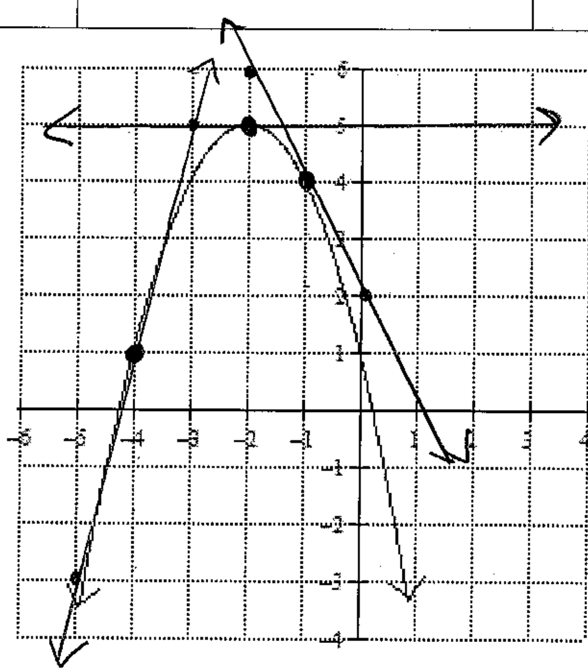
Equation of the tangent line at $x = -4$	Equation of the tangent line at $x = -2$	Equation of the tangent line at $x = -1$
• point on tangent → $f(-4) = -(-4)^2 - 4(-4) + 1 = 1$ $(-4, 1)$ • slope of tangent → $f'(-4) = -2(-4) - 4 = 4$	• P.O.T → $f(-2) = 5$ $(-2, 5)$ • S.O.T → $f'(-2) = 0$ $y - 5 = 0(x + 2)$ $\boxed{y = 5}$	• P.O.T → $f(-1) = 4$ $(-1, 4)$ • S.O.T → $f'(-1) = -2$ $\boxed{y - 4 = -2(x + 1)}$ $\boxed{y = -2x + 2}$

$y - y_1 = m(x - x_1)$

$\boxed{y - 1 = 4(x + 4)}$

or

$\boxed{y = 4x + 17}$



When you hear "**DERIVATIVE,**" you think "**SLOPE OF THE TANGENT LINE.**"

When you hear "**SLOPE OF THE TANGENT LINE,**" you think "**DERIVATIVE.**"

Example 2: Find $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ for the functions given below to find and use $f'(x)$.

$f(x)=\sqrt{x+2}$

$$\lim_{h\to 0} \frac{\sqrt{x+h+2}-\sqrt{x+2}}{h}$$

$$\frac{(\sqrt{x+h+2}-\sqrt{x+2})}{(h)} \cdot \frac{(\sqrt{x+h+2}+\sqrt{x+2})}{(\sqrt{x+h+2}+\sqrt{x+2})}$$

$$= \frac{\cancel{x}+h+\cancel{2}-\cancel{x}-\cancel{2}}{(h)(\sqrt{x+h+2}+\sqrt{x+2})} = \frac{\cancel{h}}{(\cancel{h})(\sqrt{x+h+2}+\sqrt{x+2})}$$

$$\lim_{h\to 0} \frac{1}{\sqrt{x+h+2}+\sqrt{x+2}} = \frac{1}{\sqrt{x+0+2}+\sqrt{x+2}}$$

$$\boxed{f'(x)=\frac{1}{2\sqrt{x+2}}}$$

$f(x)=\frac{3}{x+2}$

$$\lim_{h\to 0} \frac{\frac{3}{x+h+2}-\frac{3}{x+2}}{h}$$

$$\frac{3(x+2)}{(x+h+2)(x+2)} + \frac{-3\,(x+h+2)}{(x+2)(x+h+2)}$$

$$= \frac{\cancel{3x}+\cancel{6}-\cancel{3x}-3h-\cancel{6}}{(x+h+2)(x+2)} = \frac{-3h}{(x+h+2)(x+2)}$$

$$\frac{\frac{-3h}{(x+h+2)(x+2)}}{h} = \frac{-3\cancel{h}}{(x+h+2)(x+2)} \cdot \frac{1}{\cancel{h}}$$

$$\lim_{h\to 0} \frac{-3}{(x+h+2)(x+2)} = \frac{-3}{(x+0+2)(x+2)}$$

$$\boxed{f'(x) = \frac{-3}{(x+2)^2}}$$

Find the equation of the line tangent to the graph of $f(x)=\sqrt{x+2}$ at $x=7$.

• P.O.T → $f(7)=\sqrt{7+2}=\sqrt{9}=3$

$(7,3)$

• S.O.T → $f'(7)=\frac{1}{2\sqrt{7+2}}=\frac{1}{2(3)}=\frac{1}{6}$

$$\boxed{y-3=\frac{1}{6}(x-7)}$$

Find the equation of the line tangent to the graph of $f(x)=\frac{3}{x+2}$ at $x=1$.

• P.O.T → $f(1)=\frac{3}{1+2}=\frac{3}{3}=1$

$(1,1)$

• S.O.T → $f'(1)=\frac{-3}{(1+2)^2}=\frac{-3}{9}=\frac{-1}{3}$

$$\boxed{y-1=-\frac{1}{3}(x-1)}$$

Over the course of this lesson so far, you have found derivatives of several functions and evaluated that derivative at certain x – values. Look back at your work and complete the table below.

Equation of Function, $f(x)$	Equation of Derivative, $f'(x)$	Value of $f'(x)$ at the Indicated value of x	Find the Value of the Limit $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, where a is the value of x.
$f(x)=-x^2-4x+1$	$f'(x)=-2x-4$	$x=-1$ $f'(-1)=-2$ ↑ slope of the tangent at $x=-1$.	$\lim_{x\to-1}\frac{-x^2-4x+1-(-(-1)^2-4(-1)+1)}{x-(-1)}$ $\frac{-x^2-4x+1-4}{x+1}=\frac{-x^2-4x-3}{x+1}=\frac{-(x^2+4x+3)}{x+1}$ $=\frac{-(x+3)\cancel{(x+1)}}{\cancel{(x+1)}}$ $\lim_{x\to-1}-(x+3)=-(-1+3)=\boxed{-2}$
$f(x)=\sqrt{x+2}$	$f'(x)=\frac{1}{2\sqrt{x+2}}$	$x=7$ $f'(7)=1/6$ ↑ slope of tangent at $x=7$	$\lim_{x\to7}\frac{\sqrt{x+2}-(\sqrt{7+2})}{x-7}$ $\left(\frac{\sqrt{x+2}-3}{x-7}\right)\cdot\left(\frac{\sqrt{x+2}+3}{\sqrt{x+2}+3}\right)=\frac{x+2-9}{(x-7)(\sqrt{x+2}+3)}=$ $\frac{\cancel{x-7}}{\cancel{(x-7)}(\sqrt{x+2}+3)}\rightarrow\lim_{x\to7}\frac{1}{\sqrt{x+2}+3}=\frac{1}{\sqrt{7+2}+3}$

$=\frac{1}{3+3}=\boxed{1/6}$

What inference can you make that explains what the limit $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ represents?

SLOPE of the TANGENT LINE drawn to the graph of f(x) at x=a.