2017

AP® CollegeBoard

AP Calculus BC
Free-Response Questions

AP® Calculus BC Exam

SECTION II: Free Response

2017

DO NOT OPEN THIS BOOKLET OR BREAK THE SEALS ON PART B UNTIL YOU ARE TOLD TO DO SO.

At a Glance

Total Time
1 hour and 30 minutes
Number of Questions
6
Percent of Total Score
50%
Writing Instrument
Either pencil or pen with black or dark blue ink
Weight
The questions are weighted equally, but the parts of a question are not necessarily given equal weight.

Part A

Number of Questions
2
Time
30 minutes
Electronic Device
Graphing calculator required
Percent of Section II Score
33.33%

Part B

Number of Questions
4
Time
1 hour
Electronic Device
None allowed
Percent of Section II Score
66.67%

IMPORTANT Identification Information

PLEASE PRINT WITH PEN:

1. First two letters of your last name ☐☐

   First letter of your first name ☐
2. Date of birth

   ☐☐ ☐☐ ☐☐☐☐
   Month Day Year
3. Six-digit school code

   ☐☐☐☐☐☐
4. Unless I check the box below, I grant the College Board the unlimited right to use, reproduce, and publish my free-response materials, both written and oral, for educational research and instructional purposes. My name and the name of my school will not be used in any way in connection with my free-response materials. I understand that I am free to mark "No" with no effect on my score or its reporting.

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Instructions

The questions for Section II are printed in this booklet. Do not break the seals on Part B until you are told to do so. Write your solution to each part of each question in the space provided. Write clearly and legibly. Cross out any errors you make; erased or crossed-out work will not be scored.

Manage your time carefully. During Part A, work only on the questions in Part A. You are permitted to use your calculator to solve an equation, find the derivative of a function at a point, or calculate the value of a definite integral. However, you must clearly indicate the setup of your question, namely the equation, function, or integral you are using. If you use other built-in features or programs, you must show the mathematical steps necessary to produce your results. During Part B, you may continue to work on the questions in Part A without the use of a calculator.

As you begin each part, you may wish to look over the questions before starting to work on them. It is not expected that everyone will be able to complete all parts of all questions.

- Show all of your work, even though a question may not explicitly remind you to do so. Clearly label any functions, graphs, tables, or other objects that you use. Justifications require that you give mathematical reasons, and that you verify the needed conditions under which relevant theorems, properties, definitions, or tests are applied. Your work will be scored on the correctness and completeness of your methods as well as your answers. Answers without supporting work will usually not receive credit.
- Your work must be expressed in standard mathematical notation rather than calculator syntax. For example, $\int_1^5 x^2\,dx$ may not be written as fnInt(X², X, 1, 5).
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If you use decimal approximations in calculations, your work will be scored on accuracy. Unless otherwise specified, your final answers should be accurate to three places after the decimal point.
- Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

Form I
Form Code 4NBP4-S

68

AB/BC

CALCULUS BC

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

units
feet
square ft.

$V = \int$ area of cross sections

h (feet)	0	2	5	10
$A(h)$ (square feet)	50.3	14.4	6.5	2.9

1. A tank has a height of 10 feet. The area of the horizontal cross section of the tank at height h feet is given by the function A, where $A(h)$ is measured in square feet. The function A is continuous and decreases as h increases. Selected values for $A(h)$ are given in the table above.

 (a) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the volume of the tank. Indicate units of measure.

 (b) Does the approximation in part (a) overestimate or underestimate the volume of the tank? Explain your reasoning.

 (c) The area, in square feet, of the horizontal cross section at height h feet is modeled by the function f given by $f(h) = \dfrac{50.3}{e^{0.2h} + h}$. Based on this model, find the volume of the tank. Indicate units of measure.

 (d) Water is pumped into the tank. When the height of the water is 5 feet, the height is increasing at the rate of 0.26 foot per minute. Using the model from part (c), find the rate at which the volume of water is changing with respect to time when the height of the water is 5 feet. Indicate units of measure.

$\frac{dh}{dt} = 0.26$ ft/min $\qquad \frac{dV}{dt} = ?$ $\qquad h = 5$ ft.

GO ON TO THE NEXT PAGE.

$V = \int$ area of cross sections

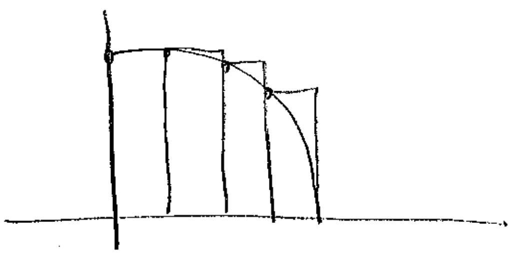
1

(a) $V = \int_0^{10} A(h)\,dh$

$(2)(50.3) + (3)(14.4) + (5)(6.5)$

$100.6 + 43.2 + 32.5 = \boxed{176 \text{ ft}^3}$

(b) A(h) is decreasing, so $\boxed{\text{overestimate}}$ using a left Riemann sum.



(c) $f(h) = \dfrac{50.3}{e^{0.2h} + h}$

$V = \underbrace{\int_0^{10} \dfrac{50.3}{e^{0.2h} + h}\,dh}_{\text{math 9}} = \boxed{101.325 \text{ ft}^3}$

(d) $V(h) = \int_0^h f(k)\,dk$

$\dfrac{dV}{dt} = f(h)\left(\dfrac{dh}{dt}\right)$

$\dfrac{dh}{dt} = 0.26 \text{ ft/min}^2$

$h = 5$ ft.

$\dfrac{dV}{dt} = f(5)(0.26)$

$= \dfrac{50.3}{e^{.2(5)} + 5}(0.26) = \boxed{1.694 \text{ ft}^3/\text{min}}$

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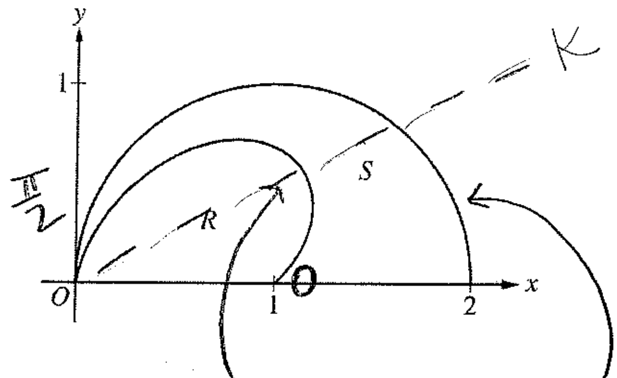
Question 1

	1 : units in parts (a), (c), and (d)
(a) Volume $= \int_0^{10} A(h)\,dh$ $\approx (2-0)\cdot A(0) + (5-2)\cdot A(2) + (10-5)\cdot A(5)$ $= 2\cdot 50.3 + 3\cdot 14.4 + 5\cdot 6.5$ $= 176.3$ cubic feet	$2: \begin{cases} 1: \text{left Riemann sum} \\ 1: \text{approximation} \end{cases}$
(b) The approximation in part (a) is an overestimate because a left Riemann sum is used and A is decreasing.	1 : overestimate with reason
(c) $\int_0^{10} f(h)\,dh = 101.325338$ The volume is 101.325 cubic feet.	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$
(d) Using the model, $V(h) = \int_0^h f(x)\,dx$. $\left.\frac{dV}{dt}\right\vert_{h=5} = \left[\frac{dV}{dh}\cdot\frac{dh}{dt}\right]_{h=5}$ $= \left[f(h)\cdot\frac{dh}{dt}\right]_{h=5}$ $= f(5)\cdot 0.26 = 1.694419$ When $h = 5$, the volume of water is changing at a rate of 1.694 cubic feet per minute.	$3: \begin{cases} 2: \frac{dV}{dt} \\ 1: \text{answer} \end{cases}$

BC

no units



2. The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

   (a) Find the area of R.

   (b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

   (c) For each θ, $0 \le \theta \le \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

   (d) Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ ? Give a reason for your answer.

END OF PART A OF SECTION II

GO ON TO THE NEXT PAGE.

(a) Area of R

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (1+\sin\theta\cos(2\theta))^2\, d\theta = \boxed{0.648}$$

(b) Area of $S = \frac{1}{2}\int [(g(\theta))^2 - (f(\theta))^2]\, d\theta$

$$\boxed{\frac{1}{2}\int_0^K [(g(\theta))^2 - (f(\theta))^2]\, d\theta = \frac{1}{2}\int_K^{\pi/2} [(g(\theta))^2 - (f(\theta))^2]\, d\theta}$$

(c) $\boxed{w(\theta) = g(\theta) - f(\theta)}$ ← distance b/t the pts $(f(\theta), \theta)$ and $(g(\theta), \theta)$

$$W_A = \text{Avg. value} = \frac{1}{\frac{\pi}{2} - 0}\underbrace{\int_0^{\pi/2} (g(\theta) - f(\theta))\, d\theta}_{\text{math 9}} = \boxed{.485}$$

(d) $w(\theta) = W_A$

$$\underbrace{g(\theta) - f(\theta)}_{Y_1} = \underbrace{0.485}_{Y_2}$$

2nd trace intersect

$\boxed{\theta = 0.518}$ ⟶ $\underbrace{w'(0.518)}_{\text{use math 8}} = -0.581$

Since $w'(0.518) < 0$, then $w(\theta)$ at $\theta = 0.518$ is decreasing

Question 2

(a) $\frac{1}{2}\int_0^{\pi/2} (f(\theta))^2\, d\theta = 0.648414$

The area of R is 0.648.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\int_0^k \left((g(\theta))^2 - (f(\theta))^2\right) d\theta = \frac{1}{2}\int_0^{\pi/2} \left((g(\theta))^2 - (f(\theta))^2\right) d\theta$

— OR —

$\int_0^k \left((g(\theta))^2 - (f(\theta))^2\right) d\theta = \int_k^{\pi/2} \left((g(\theta))^2 - (f(\theta))^2\right) d\theta$

$2 : \begin{cases} 1 : \text{integral expression for one region} \\ 1 : \text{equation} \end{cases}$

(c) $w(\theta) = g(\theta) - f(\theta)$

$$w_A = \frac{\int_0^{\pi/2} w(\theta)\, d\theta}{\frac{\pi}{2} - 0} = 0.485446$$

The average value of $w(\theta)$ on the interval $\left[0, \frac{\pi}{2}\right]$ is 0.485.

$3 : \begin{cases} 1 : w(\theta) \\ 1 : \text{integral} \\ 1 : \text{average value} \end{cases}$

(d) $w(\theta) = w_A$ for $0 \le \theta \le \frac{\pi}{2} \Rightarrow \theta = 0.517688$

$w(\theta) = w_A$ at $\theta = 0.518$ (or 0.517).

$w'(0.518) < 0 \Rightarrow w(\theta)$ is decreasing at $\theta = 0.518$.

$2 : \begin{cases} 1 : \text{solves } w(\theta) = w_A \\ 1 : \text{answer with reason} \end{cases}$

AB/BC

CALCULUS BC

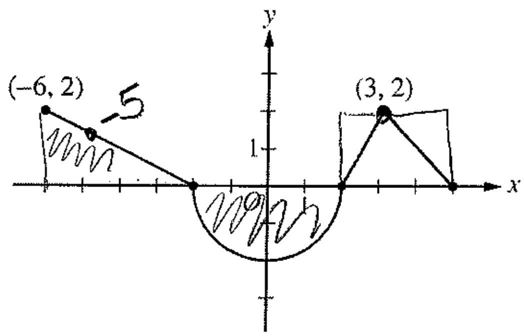
SECTION II, Part B

Time—1 hour

Number of questions—4

no units

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f'

3. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

   (a) Find the values of $f(-6)$ and $f(5)$.

   (b) On what intervals is f increasing? Justify your answer.

   (c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

   (d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

ⓐ $f(-6)$:

$$f(-2) - f(-6) = \underbrace{\int_{-6}^{-2} f'(x)\,dx}_{\text{area}}$$

$$7 - f(-6) = \tfrac{1}{2}(4)(2)$$

$$7 - f(-6) = 4$$

$$-f(-6) = -3 \quad \boxed{f(-6) = 3}$$

$f(5)$:

$$f(5) - f(-2) = \underbrace{\int_{-2}^{5} f'(x)\,dx}_{\text{area}}$$

$$f(5) - (7) = -\tfrac{1}{2}(\pi(2)^2) + \tfrac{1}{2}(3)(2)$$

$$f(5) - 7 = -2\pi + 3$$

$$\boxed{f(5) = -2\pi + 10}$$

ⓑ f increasing when $f'(x)$ is positive

$$\boxed{-6 \le x < 2 \text{ and } 2 < x < 5}$$

ⓒ ABSolute minimum on $[-6,5]$

* check endpts
* check critical pts
  ↳ $f'(x)=0$

x	$f(x)$
-6	3 (part a)
5	$-2\pi+10$ (part a)
-2	7 (given)
2	$-2\pi+7$

end pts < (-6, 5)

critical pts. < (-2, 2)

$\boxed{\text{ABSolute minimum} = -2\pi+7}$

$$f(2)-f(-2)=\underbrace{\int_{-2}^{2} f'(x)}_{\text{area}}$$

$$f(2)-7=-\tfrac{1}{2}(\pi(2)^2)$$

$$f(2)-7=-2\pi$$

$$f(2)=-2\pi+7$$

ⓓ $f''(-5)$: slope of pts $(-6,2)$ & $(-2,0)$

$$\frac{0-2}{-2-(-6)}=\frac{-2}{4}=\boxed{-1/2}$$

$f''(3)$: slope DNE b/c cusp at $x=3$.

↓

$$\lim_{x\to3^-} f''(x)=\underset{\substack{\uparrow\\ \text{slope}}}{2} \neq \lim_{x\to3^+} f''(x)=\underset{\substack{\uparrow\\ \text{slope}}}{-1}$$

Question 3

(a) $f(-6) = f(-2) + \int_{-2}^{-6} f'(x)\,dx = 7 - \int_{-6}^{-2} f'(x)\,dx = 7 - 4 = 3$

$f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx = 7 - 2\pi + 3 = 10 - 2\pi$

$3 : \begin{cases} 1 : \text{uses initial condition} \\ 1 : f(-6) \\ 1 : f(5) \end{cases}$

(b) $f'(x) > 0$ on the intervals $[-6, -2)$ and $(2, 5)$.
Therefore, f is increasing on the intervals $[-6, -2]$ and $[2, 5]$.

2 : answer with justification

(c) The absolute minimum will occur at a critical point where $f'(x) = 0$ or at an endpoint.

$f'(x) = 0 \Rightarrow x = -2,\ x = 2$

x	$f(x)$
-6	3
-2	7
2	$7 - 2\pi$
5	$10 - 2\pi$

The absolute minimum value is $f(2) = 7 - 2\pi$.

$2 : \begin{cases} 1 : \text{considers } x = 2 \\ 1 : \text{answer with justification} \end{cases}$

(d) $f''(-5) = \dfrac{2 - 0}{-6 - (-2)} = -\dfrac{1}{2}$

$\lim\limits_{x \to 3^-} \dfrac{f'(x) - f'(3)}{x - 3} = 2$ and $\lim\limits_{x \to 3^+} \dfrac{f'(x) - f'(3)}{x - 3} = -1$

$f''(3)$ does not exist because
$\lim\limits_{x \to 3^-} \dfrac{f'(x) - f'(3)}{x - 3} \neq \lim\limits_{x \to 3^+} \dfrac{f'(x) - f'(3)}{x - 3}$.

$2 : \begin{cases} 1 : f''(-5) \\ 1 : f''(3) \text{ does not exist,} \\ \quad\ \text{with explanation} \end{cases}$

AB/BC

units
°C

4. At time $t = 0$ a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\boxed{\frac{dH}{dt} = -\frac{1}{4}(H - 27)}$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

t H

X Y

dy/dx

Y

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

(a) point : $(0, 91)$

slope at $t=0$: $\frac{dH}{dt} = -\frac{1}{4}(91-27) = -\frac{1}{4}(64) = -16$

$$\boxed{H - 91 = -16(t-0)}$$

$$H - 91 = -16t$$

$$\boxed{H = -16t + 91}$$

$\underline{t=3}$: $H = -16(3) + 91$

$$= -48 + 91$$

$$= \boxed{43°C}$$

(b) $\frac{d^2H}{dt^2}$ $\qquad \frac{dH}{dt} = -\frac{1}{4}H + \frac{27}{4}$

$$\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt}$$

$$= -\frac{1}{4}\left(-\frac{1}{4}H + \frac{27}{4}\right)$$

$$\frac{d^2H}{dt^2} = \frac{1}{16}(H-27) \quad \text{or} \quad \frac{1}{16}H - \frac{27}{16}$$

$H > 27° \longrightarrow \frac{1}{16}(28-27)$

$\frac{d^2H}{dt^2} > 0 \rightarrow$ which means $H(t)$ is concave up

$\boxed{\therefore H(3) \text{ in part (a) is an underestimate}}$

(c) $\int \frac{dG}{dt} = \int -(G-27)^{2/3}$ $G(0)=91$

$$\int \frac{dG}{(G-27)^{2/3}} = \int -dt$$

$u = G-27$

$du = dG$

$$\int \frac{dG}{u^{2/3}} = \int u^{-2/3}\,dG$$

$$3u^{1/3}$$

$$\int 3(G-27)^{1/3} = -t + C \qquad G(0)=91$$

$$3(91-27)^{1/3} = -0 + C$$

$$3(64)^{1/3} = C$$

$$3(4) = C$$

$$C = 12$$

$$3(G-27)^{1/3} = -t + 12$$

$$(G-27)^{1/3} = \frac{12-t}{3}$$

$$G-27 = \left(\frac{12-t}{3}\right)^3 \rightarrow \boxed{G = \left(\frac{12-t}{3}\right)^3 + 27}$$

$\underline{t=3:}$

$$G = \left(\frac{12-3}{3}\right)^3 + 27$$

$$= \left(\frac{9}{3}\right)^3 + 27$$

$$= 27 + 27$$

$$= \boxed{54°C}$$

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Question 4

(a) $H'(0) = -\frac{1}{4}(91 - 27) = -16$
$H(0) = 91$

An equation for the tangent line is $y = 91 - 16t$.

The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius.

$$3 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$$

(b) $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$

$H > 27$ for $t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0$ for $t > 0$

Therefore, the graph of H is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.

1 : underestimate with reason

(c) $\frac{dG}{(G - 27)^{2/3}} = -dt$

$\int \frac{dG}{(G - 27)^{2/3}} = \int (-1)\, dt$

$3(G - 27)^{1/3} = -t + C$

$3(91 - 27)^{1/3} = 0 + C \Rightarrow C = 12$

$3(G - 27)^{1/3} = 12 - t$

$G(t) = 27 + \left(\frac{12 - t}{3}\right)^3$, for $0 \le t < 10$

The internal temperature of the potato at time $t = 3$ minutes is $27 + \left(\frac{12 - 3}{3}\right)^3 = 54$ degrees Celsius.

$$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and} \\ \quad\ \text{uses initial condition} \\ 1 : \text{equation involving } G \text{ and } t \\ 1 : G(t) \text{ and } G(3) \end{cases}$$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

BC no units

5. Let f be the function defined by $f(x) = \dfrac{3}{2x^2 - 7x + 5}$.

(a) Find the slope of the line tangent to the graph of f at $x = 3$.

(b) Find the x-coordinate of each critical point of f in the interval $1 < x < 2.5$. Classify each critical point as the location of a relative minimum, a relative maximum, or neither. Justify your answers.

(c) Using the identity that $\dfrac{3}{2x^2 - 7x + 5} = \dfrac{2}{2x - 5} - \dfrac{1}{x - 1}$, evaluate $\int_5^\infty f(x)\,dx$ or show that the integral diverges.

(d) Determine whether the series $\displaystyle\sum_{n=5}^{\infty} \frac{3}{2n^2 - 7n + 5}$ converges or diverges. State the conditions of the test used for determining convergence or divergence.

$$f(x) = \frac{3}{2x^2-7x+5}$$

(a) Slope of tangent at $x=3$

$$f'(x) = \frac{\cancel{(2x^2-7x+5)(0)} - (3)(4x-7)}{(2x^2-7x+5)^2} = \frac{-12x+21}{(2x^2-7x+5)^2}$$

at $x=3$: $$\frac{-12(3)+21}{(2(3)^2-7(3)+5)^2} = \boxed{\frac{-15}{4}}$$

(b) $f'(x)=0$ $$\frac{-12x+21}{(2x^2-7x+5)^2} = 0$$

$$-12x+21=0$$
$$-12x=-21$$
$$x=\frac{7}{4}$$

1 | $\frac{7}{4}$ | 2

$\frac{-12(1)+21}{+}$ | | $\frac{-12(2)+21}{+}$

$\frac{+}{+}$ pos. | | $\frac{-}{+}$ neg

$\boxed{\text{relative maximum b/c } f'(x) \text{ changes from positive to negative}}$

(c) $$\frac{3}{2x^2-7x+5} = \frac{2}{2x-5} - \frac{1}{x-1}$$

Improper Integral $$\int_5^\infty \frac{2}{2x-5} - \frac{1}{x-1}\,dx$$

$$\lim_{b\to\infty}\int_5^b \frac{2}{2x-5} - \frac{1}{x-1}$$

$$\int \frac{2}{2x-5} \quad \begin{array}{l} u=2x-5 \\ du=2 \end{array} \quad \int \frac{du}{u}$$

$$\lim_{b\to\infty}\Big[\ln(2x-5) - \ln(x-1)\Big]_5^b$$

$$\lim_{b\to\infty}\left[\ln\left(\frac{2x-5}{x-1}\right)\right]_5^b$$

$$2\cdot\frac{4}{5}$$

$$\lim_{b\to\infty}\left[\ln\left(\frac{2b-5}{b-1}\right) - \ln\left(\frac{5}{4}\right)\right]$$

$$= \ln 2 - \ln 5/4 = \ln\frac{2}{5/4} = \boxed{\ln\frac{8}{5}}$$

(d) $$\sum_{n=5}^{\infty}\frac{3}{2n^2-7n+5}$$

The series $\boxed{\text{converges}}$ by the $\boxed{\text{integral test}}$ since $\int_5^\infty \frac{3}{2x^2-7x+5}\,dx$ converges to $\ln\left(\frac{8}{5}\right)$ from part (c).

f(x) is continuous, positive, and decreasing since $f'(x) < 0$.

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Question 5

(a) $f'(x) = \dfrac{-3(4x-7)}{\left(2x^2-7x+5\right)^2}$

$f'(3) = \dfrac{(-3)(5)}{(18-21+5)^2} = -\dfrac{15}{4}$

2 : $f'(3)$

(b) $f'(x) = \dfrac{-3(4x-7)}{\left(2x^2-7x+5\right)^2} = 0 \Rightarrow x = \dfrac{7}{4}$

The only critical point in the interval $1 < x < 2.5$ has x-coordinate $\dfrac{7}{4}$.

f' changes sign from positive to negative at $x = \dfrac{7}{4}$.

Therefore, f has a relative maximum at $x = \dfrac{7}{4}$.

$2 : \begin{cases} 1 : x\text{-coordinate} \\ 1 : \text{relative maximum} \\ \quad\ \text{with justification} \end{cases}$

(c) $\displaystyle\int_5^\infty f(x)\,dx = \lim_{b\to\infty}\int_5^b \frac{3}{2x^2-7x+5}\,dx = \lim_{b\to\infty}\int_5^b \left(\frac{2}{2x-5} - \frac{1}{x-1}\right)dx$

$\displaystyle = \lim_{b\to\infty}\Big[\ln(2x-5) - \ln(x-1)\Big]_5^b = \lim_{b\to\infty}\left[\ln\left(\frac{2x-5}{x-1}\right)\right]_5^b$

$\displaystyle = \lim_{b\to\infty}\left[\ln\left(\frac{2b-5}{b-1}\right) - \ln\left(\frac{5}{4}\right)\right] = \ln 2 - \ln\left(\frac{5}{4}\right) = \ln\left(\frac{8}{5}\right)$

$3 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{limit expression} \\ 1 : \text{answer} \end{cases}$

(d) f is continuous, positive, and decreasing on $[5, \infty)$.

The series converges by the integral test since $\displaystyle\int_5^\infty \frac{3}{2x^2-7x+5}\,dx$ converges.

— OR —

$\dfrac{3}{2n^2-7n+5} > 0$ and $\dfrac{1}{n^2} > 0$ for $n \geq 5$.

Since $\displaystyle\lim_{n\to\infty}\frac{\frac{3}{2n^2-7n+5}}{\frac{1}{n^2}} = \frac{3}{2}$ and the series $\displaystyle\sum_{n=5}^\infty \frac{1}{n^2}$ converges,

the series $\displaystyle\sum_{n=5}^\infty \frac{3}{2n^2-7n+5}$ converges by the limit comparison test.

2 : answer with conditions

BC

no units

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

   (a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

   (b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

   (c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

   (d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

   $$\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < \frac{1}{500}.$$

STOP

END OF EXAM

$f(0)=0$

$f'(0)=1$

$f^{n+1}(0)=-n\cdot f^{n}(0)$

ⓐ $f(0)+\dfrac{f'(0)(x)}{1!}+\dfrac{f''(0)x^2}{2!}+\dfrac{f'''(0)x^3}{3!}+\dfrac{f^4(0)x^4}{4!}$

$n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4$

$\cancel{0}+\dfrac{1(x)}{1!}+\dfrac{-1\cdot f'(0)x^2}{2!}+\dfrac{-2\cdot f^2(0)x^3}{3!}+\dfrac{-3f^3(0)x^4}{4!}$

$x+\dfrac{(-1)(1)x^2}{2!}+\dfrac{-2(-1)x^3}{3!}+\dfrac{-3(2)x^4}{4!}$

$x+\dfrac{-x^2}{2}+\dfrac{2x^3}{6}-\dfrac{6x^4}{24}$

$$\boxed{x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}}$$

$n=1 \quad n=2 \quad n=3 \quad n=4$

general term: $\boxed{\dfrac{(-1)^{n+1}x^n}{n}}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$ at $x=1$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ Alt. series Test $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

$\frac{1}{n+1} \le \frac{1}{n}$ ✓ converges

$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ p-series test $p=1$ diverges

$\boxed{\text{converges conditionally}}$

(c) $g(x) = \int_0^x f(t)\,dt = \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \ldots\right) dt$

$$\left[\frac{t^2}{2} - \frac{t^3}{2\cdot 3} + \frac{t^4}{3\cdot 4} - \frac{t^5}{4\cdot 5} + \ldots\right]_0^x$$

$$\boxed{\frac{x^2}{2} - \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} - \frac{x^5}{4\cdot 5} + \ldots}$$

$n=1$ $\quad n=2$ $\quad n=3$ $\quad n=4$

$\boxed{\text{general term: } \frac{x^{n+1}(-1)^{n+1}}{(n+1)(n)}}$

$P_4(1/2)$ using g from part (c)

(d) Alternating Series error bound will be the absolute value of the next unused term. (which is the 5th-degree term)

$$\left|\frac{-x^5}{4.5}\right| = \left|-\frac{\left(\frac{1}{2}\right)^5}{4.5}\right| = \frac{\frac{1}{32}}{20} = \frac{1}{32}\cdot\frac{1}{20}$$

$$= \boxed{\frac{1}{640} < \frac{1}{500}} \checkmark$$

Question 6

(a) $f(0) = 0$ $f'(0) = 1$ $f''(0) = -1(1) = -1$ $f'''(0) = -2(-1) = 2$ $f^{(4)}(0) = -3(2) = -6$ The first four nonzero terms are $0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$ The general term is $\frac{(-1)^{n+1}x^n}{n}$.	$3 : \begin{cases} 1 : f''(0),\ f'''(0), \text{ and } f^{(4)}(0) \\ 1 : \text{verify terms} \\ 1 : \text{general term} \end{cases}$
(b) For $x = 1$, the Maclaurin series becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. The series does not converge absolutely because the harmonic series diverges. The series alternates with terms that decrease in magnitude to 0, and therefore the series converges conditionally.	2 : converges conditionally with reason
(c) $\int_0^x f(t)\,dt = \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1}t^n}{n} + \cdots\right) dt$ $= \left[\frac{t^2}{2} - \frac{t^3}{3\cdot 2} + \frac{t^4}{4\cdot 3} - \frac{t^5}{5\cdot 4} + \cdots + \frac{(-1)^{n+1}t^{n+1}}{(n+1)n} + \cdots\right]_{t=0}^{t=x}$ $= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1}x^{n+1}}{(n+1)n} + \cdots$	$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$
(d) The terms alternate in sign and decrease in magnitude to 0. By the alternating series error bound, the error $\left\lvert P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right\rvert$ is bounded by the magnitude of the first unused term, $\left\lvert -\frac{(1/2)^5}{20}\right\rvert$. Thus, $\left\lvert P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right\rvert \le \left\lvert -\frac{(1/2)^5}{20}\right\rvert = \frac{1}{32\cdot 20} < \frac{1}{500}$.	1 : error bound