AB/BC

units:
$A(t)$ pounds
t days

CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

1. Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.
   (a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.
   (b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.
   (c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.
   (d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

(a) Average Rate of change $= \dfrac{1}{b-a}\displaystyle\int_a^b A'(t)\,dt$

$= \dfrac{1}{30-0}\displaystyle\int_0^{30} A'(t)\,dt = \dfrac{1}{30}\left[A(30) - A(0)\right]$

$= \dfrac{1}{30}\left[0.783 - 6.687\right] = \boxed{-0.197 \text{ lbs/day}}$

(b) $A'(15)$ use math 8 on calc.

math 8 → $\dfrac{d}{dx}\left(6.687(.931)^t\right)_{x=15} = \boxed{-0.164 \text{ lb/day}}$

Derivative → This means the amount of grass clippings in the bin is decreasing at a rate of 0.164 pounds per day when $t = 15$ days.

GO ON TO THE NEXT PAGE.

ⓒ time = ?

$$\underbrace{\text{Amount of Grass Clippings}} = \underbrace{\text{Avg. Amount of Grass Clippings}}$$

$$A(t) = \frac{1}{30-0}\int_0^{30} A(t)\,dt$$

$$\underbrace{6.687(.931)^t}_{y_1} = \underbrace{\frac{1}{30}\int_0^{30} 6.687(.931)^t\,dt}_{y_2}$$

$\boxed{2nd}$ $\boxed{Trace}$ $\boxed{intersect}$ } Graphs really slow on calc.

$$\boxed{t = 12.415 \text{ days}}$$

ⓓ Lineraization = Tangent Line Approximation

$t = 30$ days

$A(30) = 6.687(.931)^{30} = 0.783$

point (30, 0.783)

Slope : $A'(30)$ at $t = 30$ → -0.056
use Math 8

$y - .783 = -.056(x - 30)$

$y - .783 = -.056x + 1.68$

$L(t) = -.056x + 2.463$ Tangent line approx.

$L(t) = 0.5$
↓

$0.5 = -.056x + 2.463$

$-1.963 = -.056x$

$x = \boxed{35.054 \text{ days}}$

Question 1

Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

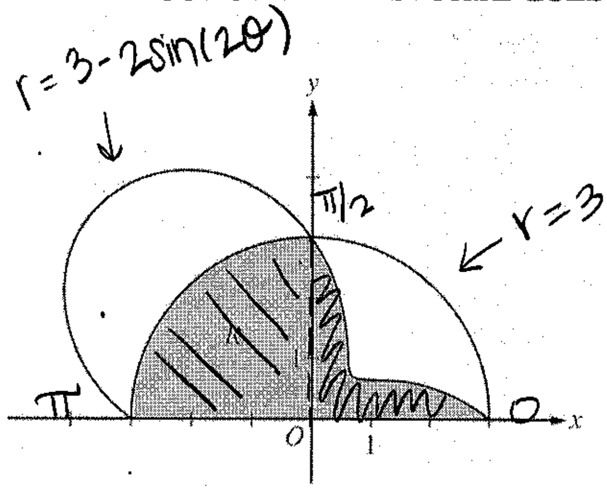
(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

(a) $\dfrac{A(30) - A(0)}{30 - 0} = -0.197$ (or -0.196) lbs/day	1 : answer with units
(b) $A'(15) = -0.164$ (or -0.163) The amount of grass clippings in the bin is decreasing at a rate of 0.164 (or 0.163) lbs/day at time $t = 15$ days.	$2 : \begin{cases} 1 : A'(15) \\ 1 : \text{interpretation} \end{cases}$
(c) $A(t) = \dfrac{1}{30}\displaystyle\int_0^{30} A(t)\,dt \Rightarrow t = 12.415$ (or 12.414)	$2 : \begin{cases} 1 : \dfrac{1}{30}\displaystyle\int_0^{30} A(t)\,dt \\ 1 : \text{answer} \end{cases}$
(d) $L(t) = A(30) + A'(30) \cdot (t - 30)$ $A'(30) = -0.055976$ $A(30) = 0.782928$ $L(t) = 0.5 \Rightarrow t = 35.054$	$4 : \begin{cases} 2 : \text{expression for } L(t) \\ 1 : L(t) = 0.5 \\ 1 : \text{answer} \end{cases}$

BC



no units

2. The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.

   (a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the $\boxed{\text{area}}$ of R.

   (b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$.

   (c) The distance between the two curves changes for $0 < \theta < \frac{\pi}{2}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{3}$.

   (d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 3$ for all times $t \ge 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$.

(a) AREA

$$\frac{1}{2}\int_0^{\pi/2} (3-2\sin 2\theta)^2\, d\theta = 2.639$$

$$+$$

$$\frac{1}{2}\int_{\pi/2}^{\pi} (3)^2\, d\theta = 7.069 \quad = \boxed{9.708}$$

END OF PART A OF SECTION II

(b) $r = 3-2\sin(2\theta)$

$x = r\cos\theta = (3-2\sin 2\theta)\cos\theta$

Find $\frac{dx}{d\theta}$ at $\theta = \pi/6$ $\longrightarrow$ use Math 8 $= \boxed{-2.366}$

GO ON TO THE NEXT PAGE.

(c) Distance b/t the two curves on $(0, \pi/2)$ is

$3-(3-2\sin 2\theta)$

rate is $\frac{d}{d\theta}(3-(3-2\sin 2\theta))$ at $\pi/3$

use Math 8 $= -1.99999 = \boxed{-2.000}$

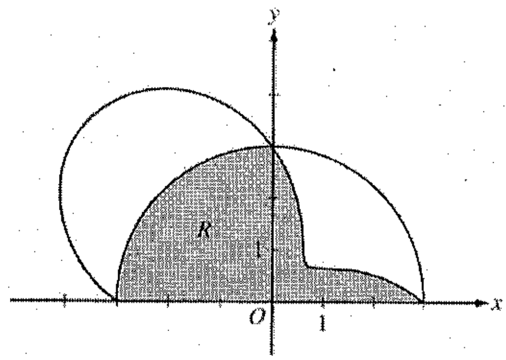
(d) $r=3-2\sin(2\theta)$ and $\frac{d\theta}{dt}=3$

$\frac{dr}{dt} = -4\cos(2\theta)\frac{d\theta}{dt}$

at $\theta=\pi/6 \rightarrow -4\cos(2\cdot\pi/6)(3) = \boxed{-6}$

Question 2

The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.



(a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the area of R.

(b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$.

(c) The distance between the two curves changes for $0 < \theta < \frac{\pi}{2}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{3}$.

(d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 3$ for all times $t \ge 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$.

(a) $$\text{Area} = \frac{9\pi}{4} + \frac{1}{2}\int_0^{\pi/2} (3 - 2\sin(2\theta))^2 \, d\theta = 9.708 \text{ (or } 9.707\text{)}$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

(b) $x = (3 - 2\sin(2\theta))\cos\theta$

$$\left.\frac{dx}{d\theta}\right|_{\theta=\pi/6} = -2.366$$

$2 : \begin{cases} 1 : \text{expression for } x \\ 1 : \text{answer} \end{cases}$

(c) The distance between the two curves is $D = 3 - (3 - 2\sin(2\theta)) = 2\sin(2\theta)$.

$$\left.\frac{dD}{d\theta}\right|_{\theta=\pi/3} = -2$$

$2 : \begin{cases} 1 : \text{expression for distance} \\ 1 : \text{answer} \end{cases}$

(d) $$\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \frac{dr}{d\theta} \cdot 3$$

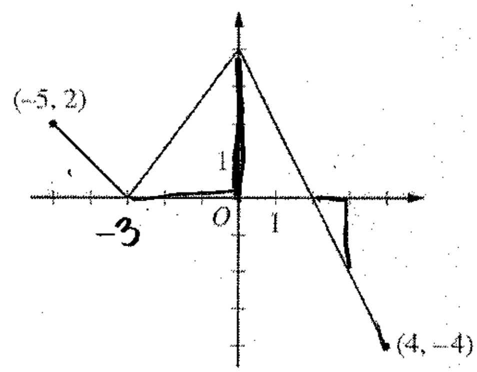
$$\left.\frac{dr}{dt}\right|_{\theta=\pi/6} = (-2)(3) = -6$$

$2 : \begin{cases} 1 : \text{chain rule with respect to } t \\ 1 : \text{answer} \end{cases}$

AB/BC

CALCULUS BC
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.



Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

   (a) Find $g(3)$.

   (b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

   (c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

   (d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

ⓐ $\underbrace{g(3) = \int_{-3}^{3} f(t)\,dt}_{\text{Area}}$

4, 3   4, 2   1, 2 below

$\frac{1}{2}(3)(4) = 6$ $\frac{1}{2}(2)(4) = 4$ $\frac{1}{2}(1)(2) = -1$

$6 + 4 + -1 = \boxed{9}$

(b) g is increasing & concave down

- g is increasing when $g'(x)$ or $f(x) > 0 \Rightarrow$
  $-5 < x < 2$

- g is concave down when $g''(x)$ or $f'(x) < 0 \Rightarrow$
  $g'(x)$ or $f(x)$ is decreasing $\Rightarrow -5 < x < -3$
  and $0 < x < 4$

Both occur when $\boxed{-5 < x < -3 \ \& \ 0 < x < 2}$

(c) $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$

$$h'(x) = \frac{(5x)g'(x) - g(x)(5)}{(5x)^2} = \frac{5(3)g'(3) - g(3)(5)}{(5\cdot 3)^2}$$

$$= \frac{15(-2) - (9)(5)}{(15)^2}$$

$g'(3) = \underbrace{f(3) = -2}_{\text{Look @ graph}}$

$$= \frac{-30-45}{225} = \boxed{\frac{-75}{225} = -\frac{1}{3}}$$

$g(3) = 9$ (from part a)

(d) $p(x) = f(x^2 - x)$ find slope of tangent line to p at $x = -1$

$p'(x) = f'(x^2 - x)(2x - 1)$

$p'(-1) = f'((-1)^2 - (-1))(2(-1) - 1) = \underbrace{f'(2)}_{\text{slope of } f @ 2}(-3)$

$$\frac{-4-4}{4-0} = (-2)(-3) = \boxed{6}$$

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Question 3

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above.

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(a) Find $g(3)$.

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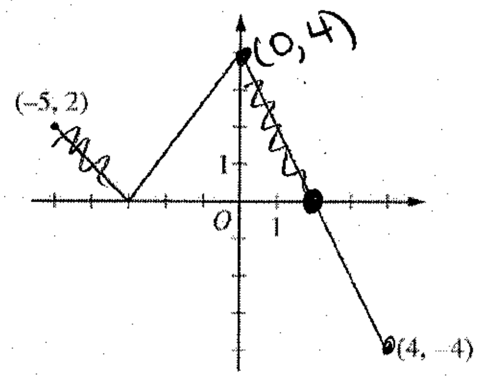
(d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.



Graph of f

(a) $g(3) = \int_{-3}^{3} f(t)\,dt = 6 + 4 - 1 = 9$

1 : answer

(b) $g'(x) = f(x)$

The graph of g is increasing and concave down on the intervals $-5 < x < -3$ and $0 < x < 2$ because $g' = f$ is positive and decreasing on these intervals.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(c) $h'(x) = \dfrac{5xg'(x) - g(x)5}{(5x)^2} = \dfrac{5xg'(x) - 5g(x)}{25x^2}$

$$h'(3) = \frac{(5)(3)g'(3) - 5g(3)}{25 \cdot 3^2}$$
$$= \frac{15(-2) - 5(9)}{225} = \frac{-75}{225} = -\frac{1}{3}$$

$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

(d) $p'(x) = f'\left(x^2 - x\right)(2x - 1)$

$p'(-1) = f'(2)(-3) = (-2)(-3) = 6$

$3 : \begin{cases} 2 : p'(x) \\ 1 : \text{answer} \end{cases}$

AB/BC

units
t = minutes
meters

velocity →

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	−120	−150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

(a) Average Acceleration $\frac{1}{b-a}\int_a^b a(t)\,dt = \frac{1}{b-a}\int_a^b v'(t)\,dt$

$$\frac{1}{8-2}\int_2^8 v'(t)\,dt = \frac{1}{6}\left[v(8) - v(2)\right]$$

$$\frac{1}{6}\left[-120 - 100\right]$$

$$\frac{1}{6}(-220) = \boxed{-\frac{110}{3}\ \text{m/min}^2}$$

(b) V_A is differentiable $\Rightarrow$ V_A is continuous.

$V_A(5) = 40$ m/min & $V_A(8) = -120$ m/min

Since V_A is continuous & $V_A(8) < -100 < V_A(5)$, then the Intermediate Value Theorem guarantees there is a velocity of -100 m/min b/t $5 < t < 8$.

(c) $t=2 \rightarrow$ 300 m east of origin station & moving east

$$\int_2^{12} V_A(t)\,dt = S_A(12) - S_A(2)$$ S = position

$$S_A(12) = S_A(2) + \int_2^{12} V_A(t)\,dt$$

$$\boxed{S_A(12) = 300 + \int_2^{12} V_A(t)\,dt}$$

Trapezoidal sum for $t=12$

$$300 + \int_2^{12} V_A(t)\,dt$$

$$\boxed{300 + \tfrac{1}{2}(5-2)(100+40) + \tfrac{1}{2}(8-5)(40+-120) + \tfrac{1}{2}(12-8)(-150+-120)}$$

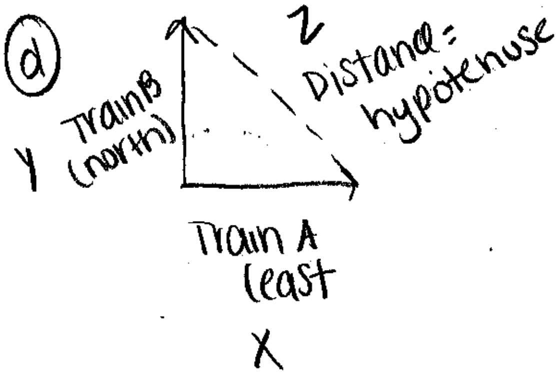
$$300 + \tfrac{1}{2}(3)(140) + \tfrac{1}{2}(3)(-80) + \tfrac{1}{2}(4)(-270)$$

$$300 + 210 + -120 + -540$$

$$= \boxed{-150}$$

The position of Train A at $t=12$ min is approx. 150 meters west of origin station.

(d)



$x^2 + y^2 = z^2$

$(300)^2 + (400)^2 = z^2$

$z = 500$

$\frac{dx}{dt} = 100 \text{ @ } t = 2$

$\frac{dy}{dt} = -5(2)^2 + 60(2) + 25$

$-20 + 120 + 25$

$= 125$

$x = 300,\ y = 400$

$x^2 + y^2 = z^2$

$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$

$2(300)(100) + 2(400)(125) = 2(500) \frac{dz}{dt}$

$60{,}000 + 100{,}000 = 1{,}000 \frac{dz}{dt}$

$160{,}000 = 1{,}000 \frac{dz}{dt}$

$\boxed{\frac{dz}{dt} = 160 \text{ m/min}}$

Question 4

Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	−120	−150

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

(a) average accel $= \dfrac{v_A(8) - v_A(2)}{8 - 2} = \dfrac{-120 - 100}{6} = -\dfrac{110}{3}$ m/min^2

1 : average acceleration

(b) v_A is differentiable $\Rightarrow$ v_A is continuous

$v_A(8) = -120 < -100 < 40 = v_A(5)$

Therefore, by the Intermediate Value Theorem, there is a time t, $5 < t < 8$, such that $v_A(t) = -100$.

$2 : \begin{cases} 1 : v_A(8) < -100 < v_A(5) \\ 1 : \text{conclusion, using IVT} \end{cases}$

(c) $s_A(12) = s_A(2) + \int_2^{12} v_A(t)\,dt = 300 + \int_2^{12} v_A(t)\,dt$

$$\int_2^{12} v_A(t)\,dt \approx 3 \cdot \frac{100 + 40}{2} + 3 \cdot \frac{40 - 120}{2} + 4 \cdot \frac{-120 - 150}{2} = -450$$

$s_A(12) \approx 300 - 450 = -150$

The position of Train A at time $t = 12$ minutes is approximately 150 meters west of Origin Station.

$3 : \begin{cases} 1 : \text{position expression} \\ 1 : \text{trapezoidal sum} \\ 1 : \text{position at time } t = 12 \end{cases}$

(d) Let x be train A's position, y train B's position, and z the distance between train A and train B.

$z^2 = x^2 + y^2 \Rightarrow 2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$

$x = 300,\ y = 400 \Rightarrow z = 500$

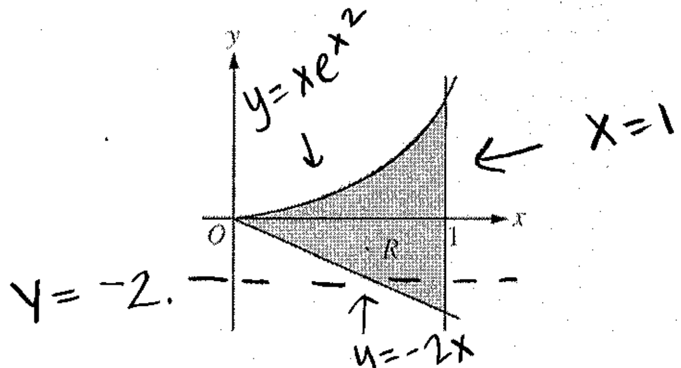
$v_B(2) = -20 + 120 + 25 = 125$

$500\dfrac{dz}{dt} = (300)(100) + (400)(125)$

$\dfrac{dz}{dt} = \dfrac{80000}{500} = 160$ meters per minute

$3 : \begin{cases} 2 : \text{implicit differentiation of distance relationship} \\ 1 : \text{answer} \end{cases}$

(BC)



5. Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

   (a) Find the area of R.

   (b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

   (c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

(a) Area R $\int_0^1 (xe^{x^2} - (-2x))dx = \int_0^1 \underbrace{xe^{x^2}} + 2x\, dx$

$u = x^2$
$du = 2x dx$
$\frac{1}{2}\int 2x e^{x^2}$
$\frac{1}{2}e^{x^2}$

$\frac{1}{2}e^{x^2} + x^2 \Big]_0^1$

$\left[\frac{1}{2}e^{1^2} + (1)^2\right] - \left[\frac{1}{2}e^{0^2} + 0^2\right]$

$\frac{1}{2}e + 1 - \frac{1}{2} + 0 = \boxed{\frac{1}{2}e + \frac{1}{2}}$

(b) volume rotated about $y = -2$ $\boxed{\pi\int_0^1 \left[(xex^2 - (-2))^2 - (-2x - (-2))^2\right]dx}$

closest to $y = -2$

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ⓒ Perimeter of R (arc length)

$$\int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$$\boxed{\int_0^1 \sqrt{1+(e^{x^2}+2x^2e^{x^2})^2}\,dx + \int_0^1 \sqrt{1+(-2)^2}\,dx + e + 2}$$

$y = xe^{x^2}$

$$\underbrace{\frac{dy}{dx} = (1)(e^{x^2}) + (x)(e^{x^2})(2x)}_{\text{arc}}$$

$y = -2x$

$$\underbrace{\frac{dy}{dx} = -2dx}_{\text{Line}}$$

vertical line from xe^{x^2} at 1 & $-2x$ at 1

$(1)(e^{1^2})$ $\quad$ $-2(1)$

$$\underbrace{e \quad \& \quad -2}_{\text{length} = e-(-2)}$$

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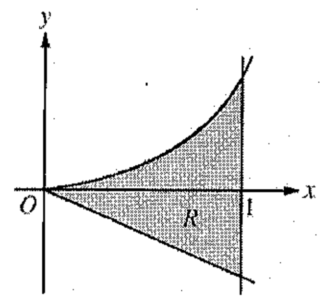
Question 5

Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.



(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

(a)
$$\text{Area} = \int_0^1 \left(xe^{x^2} - (-2x)\right) dx = \left[\frac{1}{2}e^{x^2} + x^2\right]_{x=0}^{x=1} = \left(\frac{1}{2}e + 1\right) - \frac{1}{2} = \frac{e+1}{2}$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b)
$$\text{Volume} = \pi \int_0^1 \left[\left(xe^{x^2} + 2\right)^2 - (-2x + 2)^2\right] dx$$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$

(c)
$$y' = \frac{d}{dx}\left(xe^{x^2}\right) = e^{x^2} + 2x^2e^{x^2} = e^{x^2}\left(1 + 2x^2\right)$$

$$\text{Perimeter} = \sqrt{5} + 2 + e + \int_0^1 \sqrt{1 + \left[e^{x^2}\left(1 + 2x^2\right)\right]^2}\, dx$$

$3 : \begin{cases} 1 : y' = e^{x^2}\left(1 + 2x^2\right) \\ 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

BC

6. The Taylor series for a function f about $x = 1$ is given by $\boxed{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n}$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

   (a) Find the value of R.

   (b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

   (c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

(a) Ratio Test: $\lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+2}}\,2^{n+1}(x-1)^{\cancel{n+1}}}{(n+1)}\cdot\frac{n}{\cancel{(x-1)^n}\,\cancel{2^n}\,\cancel{(-1)^{n+1}}}\right|$

$\lim_{n\to\infty}\left|\frac{2(x-1)(n)}{(n+1)}\right| = 2(x-1)\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = |2(x-1)| < 1$

$x-1 < \frac{1}{2}$

$\boxed{R = \frac{1}{2}}$

STOP

END OF EXAM

(b) $\underline{n=1}$: $(-1)^{1+1}\frac{2^1}{1}(x-1)^1 = (1)2(x-1) = \frac{2(x-1)}{1}$

$\underline{n=2}$: $(-1)^{2+1}\frac{2^2}{2}(x-1)^2 = \frac{(-1)2^2(x-1)^2}{2} = -\frac{2^2(x-1)^2}{2}$

$\underline{n=3}$: $(-1)^{3+1}\frac{2^3(x-1)^3}{3} = \frac{(1)2^3(x-1)^3}{3} = \frac{2^3(x-1)^3}{3}$

$\boxed{f'(x) = \underset{n=1}{2} - \underset{n=2}{2^2(x-1)} + \underset{n=3}{2^3(x-1)^2} + \dots}$

general term of $f'(x) =$

$\boxed{\sum_{n=1}^{\infty}(-1)^{n+1}(2)^n(x-1)^{n-1}}$

ⓒ $f'(x)$ is a geometric series

$$f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1} 2^n (x-1)^{n-1}$$

$$a_1 = (-1)^{1+1} 2^1 (x-1)^{1-1}$$
$$= (1)(2)(x-1)^0$$
$$= 2$$

$$f' = \frac{a_1}{1-r} \qquad a_1 = 2$$
$$r = -2(x-1)$$

$$f' = \frac{2}{1-(-2(x-1))} = \frac{2}{1-(-2x+2)} = \frac{2}{1+2x-2} = \boxed{\frac{2}{2x-1}}$$

$$f'(x) = \frac{2}{2x-1}$$

$$f(x) = \int \frac{2}{2x-1}\,dx \qquad u = 2x-1 \quad du = 2dx \qquad \int \frac{du}{u}$$

$$\ln|2x-1| + C$$

Since $f(1) = 0 \leftarrow (2(1-1))^1 = 0 \rightarrow$ (original series)

$$\ln|2(1)-1| + C = 0$$
$$\ln 1 + C = 0$$
$$0 + C = 0$$
$$C = 0$$

$$\boxed{f(x) = \ln|2x-1|}$$

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Question 6

The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

(a) Let a_n be the nth term of the Taylor series.

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+2}2^{n+1}(x-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n+1}2^n(x-1)^n}$$
$$= \frac{-2n(x-1)}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{-2n(x-1)}{n+1}\right| = 2|x-1|$$

$$2|x-1| < 1 \Rightarrow |x-1| < \frac{1}{2}$$

The radius of convergence is $R = \frac{1}{2}$.

$$3 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \end{cases}$$

(b) The first three nonzero terms are $2 - 4(x-1) + 8(x-1)^2$.

The general term is $(-1)^{n+1}2^n(x-1)^{n-1}$ for $n \geq 1$.

$$3 : \begin{cases} 2 : \text{first three nonzero terms} \\ 1 : \text{general term} \end{cases}$$

(c) The common ratio is $-2(x-1)$.

$$f'(x) = \frac{2}{1-(-2(x-1))} = \frac{2}{2x-1} \text{ for } |x-1| < \frac{1}{2}$$

$$f(x) = \int \frac{2}{2x-1}\,dx = \ln|2x-1| + C$$

$f(1) = 0$

$\ln|1| + C = 0 \Rightarrow C = 0$

$f(x) = \ln|2x-1|$ for $|x-1| < \frac{1}{2}$

$$3 : \begin{cases} 1 : f'(x) \\ 1 : \text{antiderivative} \\ 1 : f(x) \end{cases}$$