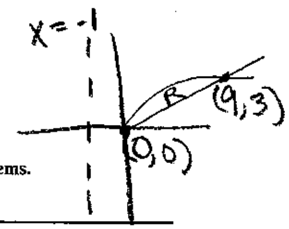
CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Let R be the region in the first quadrant bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{3}$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the vertical line $x = -1$.

(c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the y-axis are squares. Find the volume of this solid.

ⓐ $A = \int_0^9 \left(\sqrt{x} - \frac{x}{3}\right)dx = \boxed{4.500 \text{ units}^2}$

WRITE ALL WORK IN THE EXAM BOOKLET.

ⓑ $y = \sqrt{x}$ $\quad y = \frac{x}{3}$

$y^2 = x$ $\quad 3y = x$

$V = \pi \int_0^3 [3y-(-1)]^2 - [y^2-(-1)]^2 dy = \boxed{130.062 \text{ units}^3}$

↑ y-values

ⓒ $V = \int_0^3 [y^2 - 3y]^2 dy = \boxed{8.100 \text{ units}^3}$

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2. For time $t \geq 0$ hours, let $r(t) = 120\left(1 - e^{-10t^2}\right)$ represent the speed, in kilometers per hour, at which a car travels along a straight road. The number of liters of gasoline used by the car to travel x kilometers is modeled by $g(x) = 0.05x\left(1 - e^{-x/2}\right)$.

(a) How many kilometers does the car travel during the first 2 hours? integral to get rid of hrs

(b) Find the rate of change with respect to time of the number of liters of gasoline used by the car when $t = 2$ hours. Indicate units of measure.

(c) How many liters of gasoline have been used by the car when it reaches a speed of 80 kilometers per hour?

WRITE ALL WORK IN THE EXAM BOOKLET.

ⓐ Since speed is always ⊕, the total distance would be

$\int_0^2 120(1 - e^{-10t^2})dt = \boxed{206.370 \text{ Km}}$

ⓑ rate of change of # of liters of gas/hr →

$g'(t)$. at $t = 2$

$\boxed{\text{math 8}}$ $g'(2) = 0.050$ liters / Km

speed at $t = 2 \rightarrow r(2) = 120$ Km/hr

$0.050 \frac{\text{liters}}{\cancel{\text{Km}}} \cdot 120 \frac{\cancel{\text{Km}}}{\text{hr}} = \boxed{6 \frac{\text{liters}}{\text{hr}}}$

ⓒ speed = 80 Km/hr, Find t.

$80 = 120(1 - e^{-10t^2}) \rightarrow t = 0.331$ hrs

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Distance from $t = 0$ to $t = 0.331$

$\int_0^{0.331} 120(1 - e^{-10t^2}) = 10.758$ Km

$g(10.758) = 0.05(10.758)(1 - e^{-10.758/2}) = \boxed{0.535 \text{ L}}$

Distance from the river's edge (feet)	0	8	14	22	24
Depth of the water (feet)	0	7	8	2	0

3. A scientist measures the depth of the Doe River at Picnic Point. The river is 24 feet wide at this location. The measurements are taken in a straight line perpendicular to the edge of the river. The data are shown in the table above. The velocity of the water at Picnic Point, in feet per minute, is modeled by $v(t) = 16 + 2\sin\left(\sqrt{t+10}\right)$ for $0 \le t \le 120$ minutes.

   (a) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the area of the cross section of the river at Picnic Point, in square feet. Show the computations that lead to your answer.

   (b) The volumetric flow at a location along the river is the product of the cross-sectional area and the velocity of the water at that location. Use your approximation from part (a) to estimate the average value of the volumetric flow at Picnic Point, in cubic feet per minute, from $t = 0$ to $t = 120$ minutes.

   (c) The scientist proposes the function f, given by $f(x) = 8\sin\left(\frac{\pi x}{24}\right)$, as a model for the depth of the water, in feet, at Picnic Point x feet from the river's edge. Find the area of the cross section of the river at Picnic Point based on this model.

   (d) Recall that the volumetric flow is the product of the cross-sectional area and the velocity of the water at a location. To prevent flooding, water must be diverted if the average value of the volumetric flow at Picnic Point exceeds 2100 cubic feet per minute for a 20-minute period. Using your answer from part (c), find the average value of the volumetric flow during the time interval $40 \le t \le 60$ minutes. Does this value indicate that the water must be diverted?

(a) $= \frac{1}{2}(8)[0+7] + \frac{1}{2}(6)[7+8] + \frac{1}{2}(8)[8+2] + \frac{1}{2}(2)[2+0]$

WRITE ALL WORK IN THE EXAM BOOKLET.

$= 28 + 45 + 40 + 2$

$= \boxed{115 \text{ sq.ft}}$

END OF PART A OF SECTION II

(b) avg value

$\frac{1}{120-0}\int_0^{120} 115v(t)\,dt = \frac{115}{120}\int_0^{120}\left(16+2\sin\left(\sqrt{t+10}\right)\right)dt$

$= \boxed{1807.170 \text{ ft}^3/\text{min}}$

see next page

CALCULUS AB
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

4. The functions f and g are given by $f(x) = \int_0^{3x}\sqrt{4+t^2}\,dt$ and $g(x) = f(\sin x)$.

   (a) Find $f'(x)$ and $g'(x)$.

   (b) Write an equation for the line tangent to the graph of $y = g(x)$ at $x = \pi$.

   (c) Write, but do not evaluate, an integral expression that represents the maximum value of g on the interval $0 \le x \le \pi$. Justify your answer.

(a) $f'(x) = \sqrt{4+(3x)^2}\cdot 3 = \boxed{3\sqrt{4+9x^2}}$

$g(x) = f(\sin x) = \int_0^{3\sin x}\sqrt{4+t^2}\,dt$

WRITE ALL WORK IN THE EXAM BOOKLET.

$g'(x) = \sqrt{4+(3\sin x)^2}\cdot(3\cos x)$

$= \boxed{3\cos x\sqrt{4+9\sin^2 x}}$

(b) $x = \pi$

$g(\pi) = f(\sin\pi) = f(0)$ $\quad(\pi, 0)$ point $\quad(-1, 0)$

$f(0) = \int_0^0\sqrt{4+t^2}\,dt = 0$

$g'(x) = 3\cos x\sqrt{4+9\sin^2 x} \rightarrow g'(\pi) = 3\cos\pi\sqrt{4+9(\sin\pi)^2}$

$= 3(-1)\sqrt{4+9(0)^2}$

$= -6$ slope

$\boxed{y = -6(x-\pi)}$

(c) Extreme Value Thm.
endpts or critical values

$g'(x) = 0$ when $\cos x = 0$
$\therefore x = \pi/2$

$x = 0$
$x = \pi$
$x = \pi/2$

see next page

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3 c Area

$$\int_0^{24} 8\sin\left(\frac{\pi x}{24}\right) dx = \boxed{122.231 \text{ ft}^2}$$

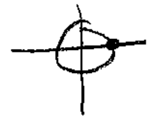
d

$$= \frac{1}{60-40}\int_{40}^{60} 122.231\left[16 + 2\sin\sqrt{t+10}\,)\right] dt$$

$$= \frac{122.231}{20}\int_{40}^{60}\left[16 + 2\sin\sqrt{t+10}\right] dt$$

$$= \boxed{2181.913 \text{ ft}^3/\text{min}}$$

Since this is greater than 2100 ft³/min, the water should be diverted.



4. (c) continued

$x=0 \longrightarrow g(0) = f(\sin 0) = f(0)$

$= \int_0^0 \sqrt{4+t^2}\,dt = 0$

$x=\pi \rightarrow g(\pi) = 0$ [from part (b)]

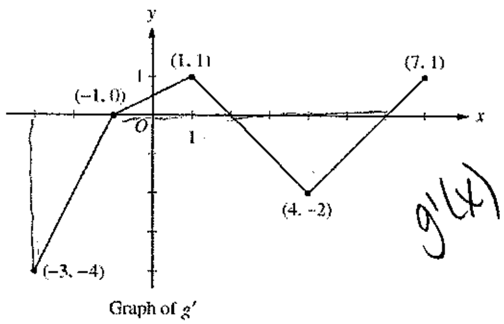
$x=\pi/2 \rightarrow g(\frac{\pi}{2}) = f(\sin(\frac{\pi}{2})) = f(1)$

$\boxed{\int_0^3 \sqrt{4+t^2}\,dt}$

↑ max must occur here

$9 - \frac{3}{2}$

$\frac{18}{2} - \frac{3}{2}$



Graph of g'

5. Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.

(a) Find the x-coordinate of all points of inflection of the graph of $y = g(x)$ for $-3 < x < 7$. Justify your answer.

(b) Find the absolute maximum value of g on the interval $-3 \le x \le 7$. Justify your answer.

(c) Find the average rate of change of $g(x)$ on the interval $-3 \le x \le 7$.

(d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

FROM part (b)

WRITE ALL WORK IN THE EXAM BOOKLET.

(c) Avg Rate of change

$$\frac{g(7) - g(-3)}{7-(-3)} = \frac{\frac{3}{2} - \frac{15}{2}}{10} = \boxed{-\frac{3}{5}}$$

(d) $$\frac{g'(7) - g'(-3)}{7-(-3)} = \frac{1-(-4)}{10} = \boxed{\frac{1}{2}}$$

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mean Value Thm
- continuous ✓
- differentiable X → $g'(x)$ is not differentiable at $x=-1$, $x=1$, & $x=4$ (cusps) $\boxed{\therefore \text{MVT does not apply}}$

(a) Points of inflection →
$g''(x)$ changes signs →
$g'(x)$ changes from incr → decr or decr → incr

$\boxed{x=1 \text{ \& } x=4}$

(b) Extreme Value Thm

endpoints : $x=-3$ $\quad g(-3) = g(2) - \int_{-3}^{2} g'(x)\,dx$ area

$= 5 - \left[\left(-\frac{1}{2}(2)(4)\right) + \left(\frac{1}{2}(3)(1)\right)\right]$

$= 5 + 4 - \frac{3}{2}$

$= \frac{15}{2}$

$x=7$ $\quad g(7) = g(2) + \int_{2}^{7} g'(x)\,dx$

$= 5 + \left(-\frac{1}{2}(4)(2)\right) + \frac{1}{2}(1)(1)$

$= 5 - 4 + \frac{1}{2}$

$= \frac{3}{2}$

rel. max : $x = 2$ $\quad g(2) = 5$

where $g'(x)$ goes from (+) → (−)

$\boxed{\text{ABSOLUTE maximum at } (-3, 15/2)}$